2.1 Factorization systems

Definition 2.1.1: $i : U \longrightarrow V$, $p : X \longrightarrow Y$ morphisms in $\mathcal{C}$.

- i has the *left lifting property* wrt p, or p has the *right lifting property* wrt i, if

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

$\forall f, g$ $\exists h$ commutative.

Notation: $i \perp p$, $i \in l(p)$, $p \in r(i)$.

Let A be a class of morphisms in $\mathcal{C}$.

Then $l(A) := \{ i : i \perp p \ \forall p \in A \}$,

$r(A) := \{ p : i \perp p \ \forall i \in A \}$.

Example: $\mathcal{C} = \text{Top}$.

Homotopy lifting property:

$$\begin{tikzcd} X \arrow[r, "\tilde{H}_0"] \arrow[d, "\mathrm{id}_X \times \{0\}"'] & Y \arrow[d, "f"] \\ X \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & Z \end{tikzcd}$$

i.e. (X, f) has the HLP iff $\forall$ homotopies $H : X \times [0,1] \longrightarrow Z$ and $\forall$ maps $\tilde{H}_0$ 'lifting' H_0, $\exists$ a homotopy $\tilde{H} : X \times [0,1] \longrightarrow Y$ 'lifting' f.

Definition 2.1.2: $X, U \in \mathcal{C}$.

X is a *retract* of U if $\exists$ diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "1_X"'] & U \arrow[r, "p"] & X. \end{tikzcd}$$

When $\mathcal{C} = \mathrm{Top}$: X is a retract of U.

When $\mathcal{C} = \mathrm{Ar}(\mathcal{D})$, category of morphisms in $\mathcal{D}$:

$f : X \longrightarrow Y$ retract of $g : U \longrightarrow V$ iff $\exists$ diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1_X"] & U \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\
Y \arrow[r] \arrow[rr, bend right, "1_Y"'] & V \arrow[r] & Y
\end{tikzcd}$$

Class of morphisms F is <u>saturated</u> if it is stable under:

i) retractions;

ii) push-outs;

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "a'"'] & Y'
\end{tikzcd}$$

push-out square.

$f \in F \Rightarrow f' \in F$.

iii) transfinite composition: $(j < i \Leftrightarrow \exists !\, j \to i)$

I a well-ordered set $\rightsquigarrow$ I a filtered category.

$X : I \longrightarrow \mathcal{C}$ a functor.

- $\forall i \in I$: $\varinjlim_{j<i} X(j)$ is representable;
- induced map $\lim\limits_{j<i} X(j) \longrightarrow X(i)$ is in F;

- canonical map $X(0) \longrightarrow \varinjlim_{i\in I} X(i)$ is in F.
  least element

Proposition 2.1.4

$f \perp f'$

a) $F \subset r(F') \Longleftrightarrow F' \subset l(F)$;

b) $F \subset F' \Longrightarrow l(F') \subset l(F)$;

c) $F \subset F' \Longrightarrow r(F') \subset r(F)$;

d) $r(F) = r(l(r(F)))$;

e) $l(F) = l(r(l(F)))$;

f) $l(F)$ saturated;

g) $r(F)$ cosaturated (saturated in $\mathcal{C}^{op}$).

Follow from definitions.

Proposition 2.1.5: Retract lemma.

Assume
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & T \arrow[ur, "p"'] & \end{tikzcd}$$
in $\mathcal{C}$.

i) If $f \in r(i)$, then f is a retract of p.
ii) If $f \in l(p)$, then f is a retract of i.

Proof ii):

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & T \arrow[d, "p"] \\ Y \arrow[r, "id_Y"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd} \Longrightarrow \begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "f"'] \arrow[rr, bend left, "id_X"] & X \arrow[r, "id_X"] \arrow[d, "i"] & X \arrow[d, "f"] \\ Y \arrow[r, "h"] \arrow[rr, bend right] & T \arrow[r, "p"] & Y \end{tikzcd}$$

$\square$

$\underbrace{\qquad\qquad}_{\mathrm{id}_Y}$

Definition 2.1.7: (A, B) couple of morphism classes of $\mathcal{C}$. Then (A, B) is a *weak factorization system* if

i) A & B stable under retracts;

ii) $A \subset l(B)$ (i.e. $B \subset r(A)$);

iii) Any $f : X \longrightarrow Y$ admits a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & T \arrow[ur, "p"'] & \end{tikzcd}$$

with $i \in A$, $p \in B$.

Note: Not just $A \subset l(B)$, but actually $A = l(B)$:

$$f \in l(B) \Rightarrow f \in l(p) \underset{\substack{\uparrow \\ \text{retract} \\ \text{lemma}}}{\Longrightarrow} f \text{ retract of } i \underset{\substack{\uparrow \\ \text{stability} \\ \text{of } A}}{\Longrightarrow} f \in A.$$

Similarly $B = r(A)$.

example: $\mathcal{C} = \mathrm{CRing}$

$A = \{\text{projective morphisms}\}$,

$B = \{\text{epimorphisms}\}$.

Corollary 2.1.10:

A a small category, I a small set of presheaves in $\hat{A}$. Then $(l(r(I)), r(I))$ is a weak factorization system in $\hat{A}$.

Proof (idea):

- Follows from *small object argument* (2.1.9):
  $\exists$ cardinal κ s.t. $\forall$ presheaves $X \in I$, the functor

$\mathrm{Hom}_{\hat{A}}(X,-):\hat{A}\longrightarrow \mathrm{Set}$ commutes with colimits indexed by κ-filtered well ordered sets.

- If $X\cong h_a$ (representable), then Yoneda lemma gives $\mathrm{Hom}_{\hat{A}}(X,Y)\cong Y_a$, so $\mathrm{Hom}_{\hat{A}}(X,-)\cong ev_a$ commutes with colimits.
- If $X\cong \coprod_{j\in J} h_{a_j}$ with $|J|<\kappa$, then $\mathrm{Hom}_{\hat{A}}(X,-)\cong \prod_{j\in J} ev_{a_j}$ commutes with κ-colimits (?).
- Proposition 1.1.8: $X=\varinjlim_{(a,s)\in A/X} h_a$ (small colimit), so X = coequaliser of maps btwn sums of h_{a_j}. Then take κ large enough.

<u>Proposition 2.1.12</u>:

$F:\mathcal{C}\rightleftarrows\mathcal{C}':G$ an adjunction, (A,B) and (A',B') w.f.s. in $\mathcal{C}$ and $\mathcal{C}'$. Then $F(A)\subset A' \iff G(B')\subset B$.

Proof: given $i\in A=L(B)$, want to show $F(i)\in A'=L(B')$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "i"'] & G(X') \arrow[d, "G(p)"] \\ Y \arrow[r, "b"'] \arrow[ur, dashed, "h"] & G(Y') \end{tikzcd} \quad \overset{\leftrightsquigarrow}{\text{adjunction}} \quad \begin{tikzcd} F(X) \arrow[r, "F(a)"] \arrow[d, "F(i)"'] & X' \arrow[d, "p"] \\ F(Y) \arrow[r, "F(b)"'] \arrow[ur, dashed, "F(h)"] & Y' \end{tikzcd}$$

<u>2.2 Model categories</u>

<u>Definition 2.2.1</u>:

A locally small category $\mathcal{C}$ is a *model category* if it has three classes of morphisms W, Fib, Cof

weak equivalences fibrations cofibequivalences fibrations rations

such that:

i) $\mathcal{C}$ has finite limits & colimits.

ii) W has 2-out-of-3 property:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

if 2 morphisms are in W, then 3^{rd} is too.

iii) Both $(Cof, Fib \cap W)$ and $(Cof \cap W, Fib)$ are weak factorization systems.

trivial fibrations

trivial cofibrations

Terminology: $\emptyset$, e are initial, final objects in $\mathcal{C}$.
The object $X \in \mathcal{C}$ is:

- *fibrant* if $X \longrightarrow e$ is a fibration;
- *cofibrant* if $\emptyset \longrightarrow X$ is a cofibration.

Remark: Axiom iii) $\Rightarrow$ every f has factorization $f = pi$ (i cofibration, p trivial fibration), since

$$Cof = l(Fib \cap W) \implies \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "p'"] \\ T \arrow[r, "id_T"'] \arrow[ur, dashed, "p"] & T \end{tikzcd}$$

Similarly, $f = qj$ (j trivial cofibration, q fibration).

<u>Important example</u> (2.1.11):

The category $\hat{}$ of presheaves over a small

The category $\hat{A}$ of presheaves over a small category A has a model category structure:

$W = \{\text{all morphisms in } \hat{A}\}$,

$\mathrm{Cof} = \{\text{monomorphisms in } \hat{A}\}$,

$\mathrm{Fib} = r(\mathrm{Cof})$.

i) finite limits & colimits;

ii) W has 2-out-of-3;

iii) show that $(\mathrm{Cof}, \mathrm{Fib})$ is a weak factorization system:

- Show $\mathrm{Cof} = l(r(I))$, where
  $I = \left\{ f : X \longrightarrow Y \;\middle|\; \begin{array}{l} f \text{ monomorphism} \\ Y \text{ quotient of representable presheaf} \end{array} \right\} \subset \hat{A}$.
- Then $(\mathrm{Cof}, \mathrm{Fib}) = (l(r(I)), r(I))$.
- Apply small object argument.

Proposition 2.2.6 (stability):

$\mathcal{C}$ a model category.

i) $\mathcal{C}^{op}$ a model category.

ii) $\forall X \in \mathcal{C}$, slice category $\mathcal{C}/X$ a model category.

Proposition 2.2.7 (Ken Brown):

$\mathcal{C}$ a model category, $\mathcal{D}$ a category with weak equivalences, $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor.

If F sends $f' : X' \longrightarrow Y' \in \mathrm{Cof} \cap W$ to $F(f') \in W$,

cofibrant objects

then F sends $f: X \longrightarrow Y \in \underline{W}$ to $F(f) \in W$.

Proof: Let $f: X \longrightarrow Y \in W$, X, Y cofibrant.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y \arrow[d, "j"] \\ X \arrow[r, "i"'] & X \amalg Y \end{tikzcd}$$

Want to show $F(f) \in W$.

i and j are cofibrations, since Cof is stable under retracts (?).

Factor $(f, 1_Y): X \amalg Y \longrightarrow Y$ into cofibration $K: X \amalg Y \longrightarrow T$ and trivial fibration $p: T \longrightarrow Y$. Then

$$(1) \quad \begin{tikzcd} X \arrow[rr, "Ki"] \arrow[dr, "f"'] & & T \arrow[dl, "p"] \\ & Y & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} Y \arrow[rr, "Kj"] \arrow[dr, "1_Y"'] & & T \arrow[dl, "p"] \\ & Y & \end{tikzcd} \quad (2)$$

- Diagram (2): $Kj \in \mathrm{Cof} \cap W$, Y and T cofibrant $\Longrightarrow F(Kj) \in W \Longrightarrow F(p) \in W$.
- Diagram (1): $Ki \in \mathrm{Cof} \cap W$, X and T cofibrant $\Longrightarrow F(Ki) \in W \Longrightarrow F(f) \in W$. $\square$

Definition 2.2.8:

Category $\mathcal{C}$ endowed with class of morphisms W. A *localization* of $\mathcal{C}$ by W is a functor:

$$\gamma: \mathcal{C} \longrightarrow ho(\mathcal{C}) \quad (R \longrightarrow S^{-1}R)$$

such that $\forall \mathcal{D}$ the functor of composition with γ

$$\gamma^*: \mathrm{Hom}(ho(\mathcal{C}), \mathcal{D}) \longrightarrow \mathrm{Hom}_W(\mathcal{C}, \mathcal{D})$$

is an equivalence.

functors sending W to isomorphisms

Same as saying that γ is universal among functors sending W to isomorphisms.

Terminology: $ho(\mathcal{C})$ is the localization of $\mathcal{C}$.

<u>Proposition 2.2.9</u>

Always $\exists$ localization $ho(\mathcal{C})$ of $\mathcal{C}$ such that γ^* is an isomorphism of categories.

Definition of $ho(\mathcal{C})$:

i) $ob(ho(\mathcal{C})) = ob(\mathcal{C})$

ii) $\mathrm{Hom}_{ho(\mathcal{C})}(X,Y)$ is equivalence classes of diagrams

$$X = X_0 \leftarrow X_1 \rightarrow X_2 \leftarrow \cdots \rightarrow X_{n-1} \leftarrow X_n \rightarrow X_{n+1} = Y \quad (n \geq 0)$$

where each $\leftarrow$ is in W or is the identity.

Equivalence relation:

$$\begin{tikzcd}
 & X_1 \arrow[r] \arrow[ld] \arrow[d] & X_2 \arrow[d] \arrow[r, phantom, "\cdots"] & X_{n-1} \arrow[d] & X_n \arrow[l] \arrow[d] \arrow[rd] & \\
X & X_1' \arrow[l] \arrow[r] & X_2' \arrow[r, phantom, "\cdots"] & X_{n-1}' & X_n' \arrow[l] \arrow[r] & Y
\end{tikzcd}$$