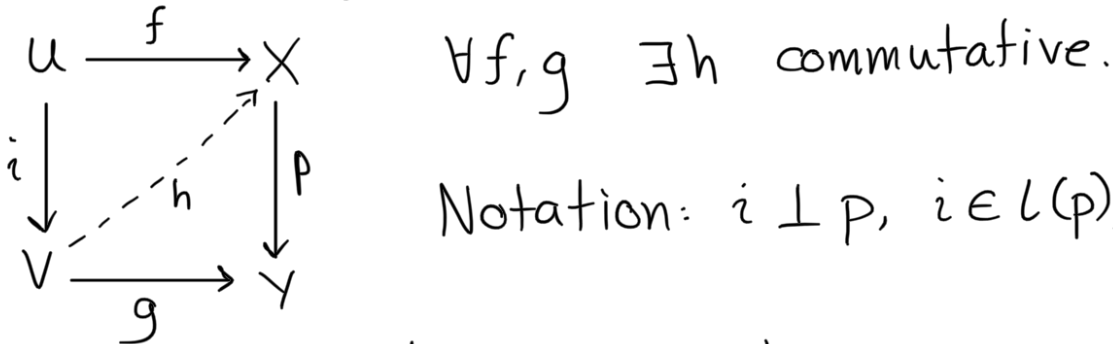


2.1 Factorization systems

Definition 2.1.1: $i: U \rightarrow V$, $p: X \rightarrow Y$ morphisms in \mathcal{C} .

- i has the left lifting property wrt p , or p has the right lifting property wrt i , if



Notation: $i \perp p$, $i \in L(p)$, $p \in r(i)$.

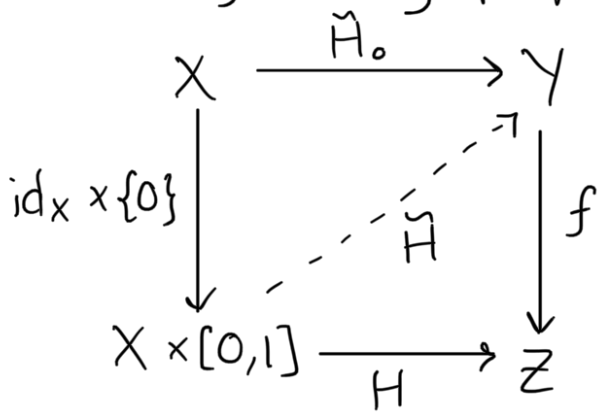
Let A be a class of morphisms in \mathcal{C} .

Then $L(A) := \{i : i \perp p \ \forall p \in A\}$,

$r(A) := \{p : i \perp p \ \forall i \in A\}$.

Example: $\mathcal{C} = \text{Top}$.

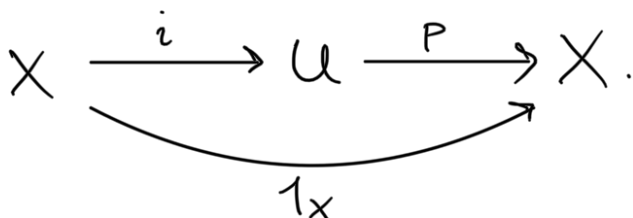
Homotopy lifting property:



i.e. (X, f) has the HLP iff \forall homotopies $H: X \times [0,1] \rightarrow Z$ and \forall maps \tilde{H}_0 'lifting' H_0 , \exists a homotopy $\tilde{H}: X \times [0,1] \rightarrow Y$ 'lifting' f .

Definition 2.1.2: $X, U \in \mathcal{C}$.

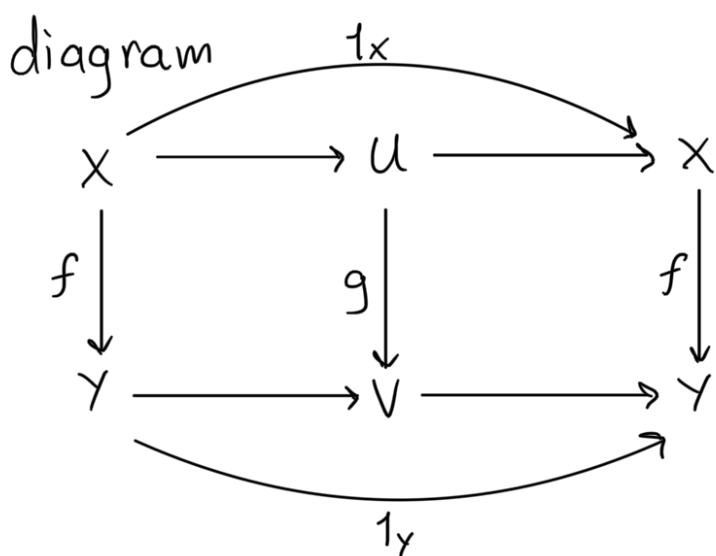
X is a retract of U if \exists diagram



When $\mathcal{C} = \text{Top}$: X is a retract of U .

When $\mathcal{C} = \text{Ar}(\mathcal{D})$, category of morphisms in \mathcal{D} :

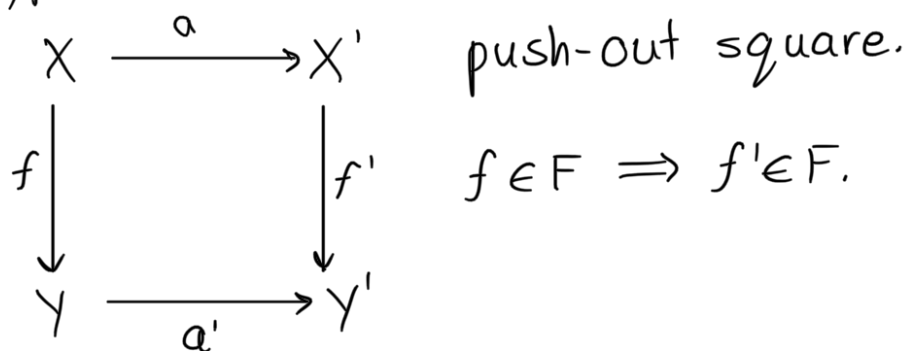
$f: X \rightarrow Y$ retract of $g: U \rightarrow V$ iff \exists



Class of morphisms F is saturated if it is stable under:

i) retractions;

ii) push-outs;



iii) transfinite composition: $(j < i \Leftrightarrow \exists! j \rightarrow i)$

I a well-ordered set $\rightsquigarrow I$ a filtered category.

$X: I \rightarrow \mathcal{C}$ a functor.

- $\forall i \in I: \lim_{j < i} X(j)$ is representable;

- induced map $\lim_{j < i} X(j) \rightarrow X(i)$ is in F ;

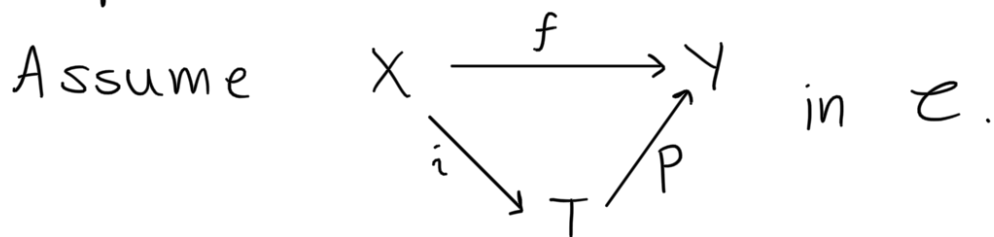
• canonical map $X(0) \xrightarrow{\text{least element}} \lim_{\rightarrow i \in I} X(i)$ is in F .

Proposition 2.1.4

- a) $F \subset r(F') \Leftrightarrow F' \subset L(F)$; $f \perp f'$
- b) $F \subset F' \Rightarrow L(F') \subset L(F)$;
- c) $F \subset F' \Rightarrow r(F') \subset r(F)$;
- d) $r(F) = r(L(r(F)))$;
- e) $L(F) = L(r(L(F)))$;
- f) $L(F)$ saturated;
- g) $r(F)$ cosaturated (saturated in \mathcal{C}^{op}).

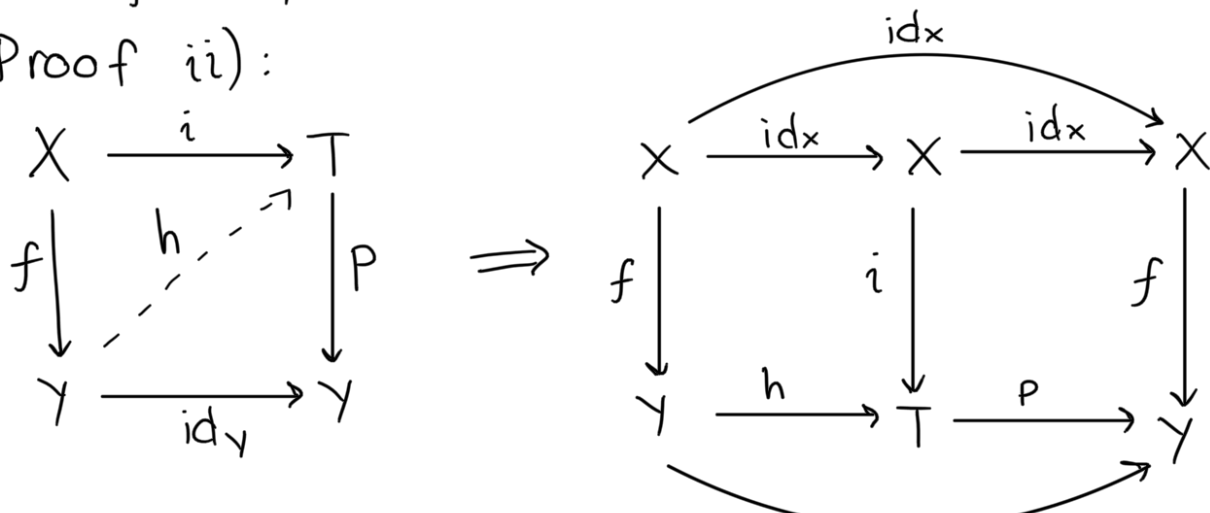
Follow from definitions.

Proposition 2.1.5: Retract lemma.



- i) If $f \in r(i)$, then f is a retract of p .
- ii) If $f \in L(p)$, then f is a retract of i .

Proof ii):



□

id_y

Definition 2.1.7: (A, B) couple of morphism classes of \mathcal{C} . Then (A, B) is a weak factorization system if

- i) A & B stable under retracts;
- ii) $A \subset l(B)$ (i.e. $B \subset r(A)$);
- iii) Any $f: X \rightarrow Y$ admits a factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow i & \nearrow p \\ & T & \end{array} \quad \text{with } i \in A, p \in B.$$

Note: Not just $A \subset l(B)$, but actually $A = l(B)$:

$$f \in l(B) \Rightarrow f \in l(p) \xRightarrow{\text{retract lemma}} f \text{ retract of } i \xRightarrow{\text{stability of } A} f \in A.$$

Similarly $B = r(A)$.

example: $\mathcal{C} = \text{CRing}$

$A = \{\text{projective morphisms}\},$

$B = \{\text{epimorphisms}\}.$

Corollary 2.1.10:

\mathcal{A} a small category, I a small set of presheaves in $\hat{\mathcal{A}}$. Then $(l(r(I)), r(I))$ is a weak factorization system in $\hat{\mathcal{A}}$.

Proof (idea):

• Follows from *small object argument* (2.1.9):

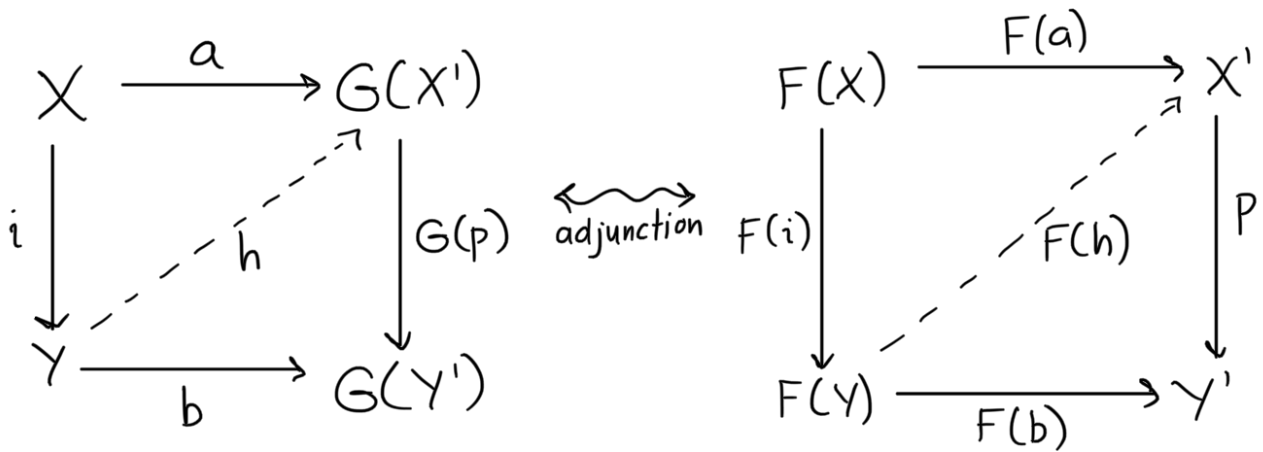
\exists cardinal κ s.t. \forall presheaves $X \in I$, the functor

$\text{Hom}_{\hat{A}}(X, -) : \hat{A} \rightarrow \text{Set}$ commutes with colimits indexed by K -filtered well ordered sets.

- If $X \cong h_a$ (representable), then Yoneda lemma gives $\text{Hom}_{\hat{A}}(X, Y) \cong Y_a$, so $\text{Hom}_{\hat{A}}(X, -) \cong \text{ev}_a$ commutes with colimits.
- If $X \cong \coprod_{j \in J} h_{a_j}$ with $|J| < K$, then $\text{Hom}_{\hat{A}}(X, -) \cong \prod_{j \in J} \text{ev}_{a_j}$ commutes with K -colimits (?).
- Proposition 1.1.8: $X = \lim_{(a,s) \in A/X} h_a$ (small colimit), so $X = \text{coequaliser of maps btwn sums of } h_{a_j}$. Then take K large enough.

Proposition 2.1.12:

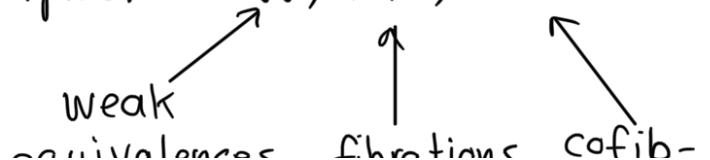
$F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$ an adjunction, (A, B) and (A', B') w.f.s. in \mathcal{C} and \mathcal{C}' . Then $F(A) \subset A' \iff G(B') \subset B$.
 Proof: given $i \in A = L(B)$, want to show $F(i) \in A' = L(B')$.



2.2 Model categories

Definition 2.2.1:

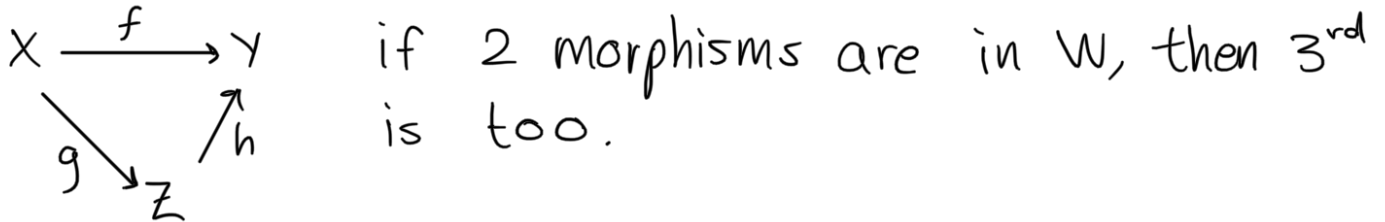
A locally small category \mathcal{C} is a model category if it has three classes of morphisms $W, \text{Fib}, \text{Cof}$



such that:

i) \mathcal{C} has finite limits & colimits.

ii) W has 2-out-of-3 property:



iii) Both $(\text{Cof}, \text{Fib} \cap W)$ and $(\text{Cof} \cap W, \text{Fib})$ are weak factorization systems.

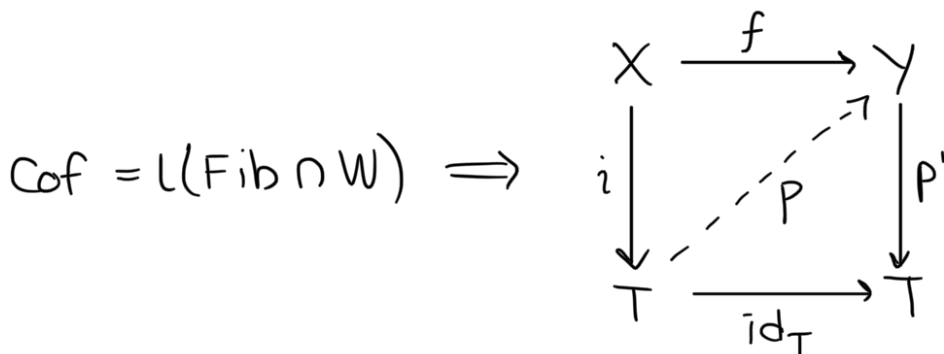
\uparrow trivial fibrations \uparrow trivial cofibrations

Terminology: \emptyset, e are initial, final objects in \mathcal{C} .

The object $X \in \mathcal{C}$ is:

- *fibrant* if $X \rightarrow e$ is a fibration;
- *cofibrant* if $\emptyset \rightarrow X$ is a cofibration.

Remark: Axiom iii) \Rightarrow every f has factorization $f = p \circ i$ (i cofibration, p trivial fibration), since



Similarly, $f = q \circ j$ (j trivial cofibration, q fibration).

Important example (2.1.11):

\rightarrow ... $\hat{\mathcal{C}}$ of ... over a small

The category \hat{A} of presheaves over a small category A has a model category structure:

$$W = \{\text{all morphisms in } \hat{A}\},$$

$$\text{Cof} = \{\text{monomorphisms in } \hat{A}\},$$

$$\text{Fib} = r(\text{Cof}).$$

i) finite limits & colimits;

ii) W has 2-out-of-3;

iii) show that (Cof, Fib) is a weak factorization system:

- Show $\text{Cof} = L(r(I))$, where

$$I = \left\{ f: X \rightarrow Y \mid \begin{array}{l} f \text{ monomorphism} \\ Y \text{ quotient of representable presheaf} \end{array} \right\} \subset \hat{A}.$$

- Then $(\text{Cof}, \text{Fib}) = (L(r(I)), r(I))$.

- Apply small object argument.

Proposition 2.2.6 (stability):

\mathcal{C} a model category.

i) \mathcal{C}^{op} a model category.

ii) $\forall X \in \mathcal{C}$, slice category \mathcal{C}/X a model category.

Proposition 2.2.7 (Ken Brown):

\mathcal{C} a model category, \mathcal{D} a category with weak equivalences, $F: \mathcal{C} \rightarrow \mathcal{D}$ a functor.

If F sends $f': X' \rightarrow Y' \in \text{Cof} \cap W$ to $F(f') \in W$,



then F sends $f: X \rightarrow Y \in \underline{W}$ to $F(f) \in W$.

Proof: Let $f: X \rightarrow Y \in W$, X, Y cofibrant.

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & & \downarrow j \\ X & \xrightarrow{i} & X \amalg Y \end{array} \quad \begin{array}{l} \text{Want to show } F(f) \in W. \\ i \text{ and } j \text{ are cofibrations, since} \\ \text{Cof is stable under retracts (?).} \end{array}$$

Factor $(f, 1_Y): X \amalg Y \rightarrow Y$ into cofibration

$k: X \amalg Y \rightarrow T$ and trivial fibration

$p: T \rightarrow Y$. Then

$$(1) \quad \begin{array}{ccc} X & \xrightarrow{k_i} & T \\ & \searrow f & \swarrow p \\ & & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \xrightarrow{k_j} & T \\ & \searrow 1_Y & \swarrow p \\ & & Y \end{array} \quad (2)$$

- Diagram (2): $k_j \in \text{Cof} \cap W$, Y and T cofibrant
 $\Rightarrow F(k_j) \in W \Rightarrow F(p) \in W$.
- Diagram (1): $k_i \in \text{Cof} \cap W$, X and T cofibrant
 $\Rightarrow F(k_i) \in W \Rightarrow F(f) \in W$. \square

Definition 2.2.8 :

Category \mathcal{C} endowed with class of morphisms W .
A localization of \mathcal{C} by W is a functor:

$$\gamma: \mathcal{C} \longrightarrow \text{ho}(\mathcal{C}) \quad (R \longrightarrow S^{-1}R)$$

such that $\forall \mathcal{D}$ the functor of composition with γ

$$\gamma^* : \text{Hom}(\text{ho}(\mathcal{C}), \mathcal{D}) \longrightarrow \text{Hom}_W(\mathcal{C}, \mathcal{D})$$

is an equivalence.

↑
functors sending
W to isomorphisms

Same as saying that γ is universal among functors sending W to isomorphisms.

Terminology: $\text{ho}(\mathcal{C})$ is the localization of \mathcal{C} .

Proposition 2.2.9

Always \exists localization $\text{ho}(\mathcal{C})$ of \mathcal{C} such that γ^* is an isomorphism of categories.

Definition of $\text{ho}(\mathcal{C})$:

i) $\text{ob}(\text{ho}(\mathcal{C})) = \text{ob}(\mathcal{C})$

ii) $\text{Hom}_{\text{ho}(\mathcal{C})}(X, Y)$ is equivalence classes of diagrams
 $X = X_0 \leftarrow X_1 \rightarrow X_2 \leftarrow \dots \rightarrow X_{n-1} \leftarrow X_n \rightarrow X_{n+1} = Y$ ($n \geq 0$)
 where each \leftarrow is in W or is the identity.

Equivalence relation:

