Higher Category Lecture Group

- Book: Higher Categories and homotopical algebra

<u>Cisinski</u>.

Why ∞-categories

. study top. space

$\rightsquigarrow$ continuous maps between these space

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Uparrow\alpha"] & Y \end{tikzcd}$$

$H : X \times [0,1] \longrightarrow Y$

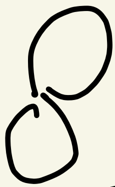
$H|_{X \times \{0\}} = f \quad H|_{X \times \{1\}} = g$

X top. space

- fundamental group,

  $x_0 \in X$

  $\pi_1(X, x_0)$ homotopy classes of loops through x_0.



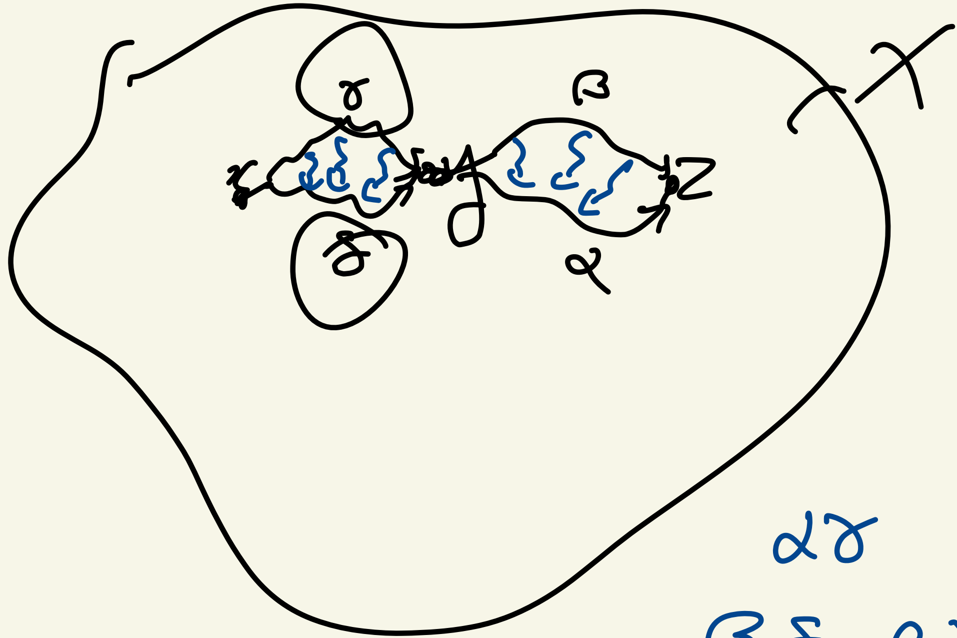
fundamental groupoid

$\Pi_1(X)$

- objects $\to$ points of X
- morphisms $\to$ homotopy classes of path

$x \rightsquigarrow y$.

$x \in X$, $\mathrm{Aut}_{\Pi_1(X)}(x) = \pi_1(X, x)$.



$\alpha\gamma$

$\beta\delta$ $\beta\gamma$

$\Pi_\infty(X)$ obj: points of X

1-morphisms: paths $x \rightsquigarrow y$

2-morphisms: homotopies between paths

3-morphisms: homotopies between homotopies

$\vdots$

$\pi_0(X), \Pi_1(X)$ $\pi_2(X)$ ..

object : Top spaces

1-morphism : C^0 maps

2-morphism : homotopies of maps $\mathbb{D}$

⋮

∞-category of Top. spaces

Last example

Top Sch/S S

$$S \left\{ \begin{array}{l} \mathrm{Top} \longrightarrow \mathrm{Cat} \\ X \longmapsto \mathrm{Sheaves}(X) \end{array} \right.$$

: category of category

presheaves(X)

Coh(X) ...

$S(X)$

$S(\emptyset) = \mathrm{Set} : \mathcal{G} \mapsto (U \mapsto \mathcal{G}(f^{-1}(U)))$

$X \xrightarrow{f} Y \xrightarrow{g} Y$

$$\begin{tikzcd}
S(X) \arrow[rr, "f_*"] \arrow[dr, "(g\circ f)_*"'] & & S(Y) \arrow[dl, "g_*"] \\
& S(Z) &
\end{tikzcd}$$

$S = K^b(S^{pre}(X))$

$\hom(\ )$

$= \hom(\ )/\sim$

homotopy of chains.

$$f_* \circ g_* \overset{?}{=} (g\circ f)_*$$

$$f_* \circ g_* \overset{\alpha}{\Rightarrow} \underline{(g\circ f)_*}$$

f_* $\quad \boxed{f^!}$

composition defined up to natural equivalence.

$$S : \mathrm{Top} \longrightarrow \mathrm{Cat}_2$$

Ob : categories

1-morphisms : functors

2-morphisms : natural transformations

2-category

composition and associativity
defined up to 2-morphisms

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "g\circ f"] \arrow[r, bend right, "(g\circ f)'"'] \arrow[r, phantom, "\Uparrow"] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "g"] \arrow[rr, bend right, "\epsilon\circ g"'] \arrow[rr, phantom, "\Uparrow\alpha"] & \mathcal{D} \arrow[r, "\epsilon"] & \mathcal{E} \end{tikzcd}$$

$(\infty, 1)$-category

$i > 1$

Stable ∞-categories. // Higher Algebra chapter 1

$D(X)$ = objects ($K_\bullet$) ~~invert~~ up to homotopy

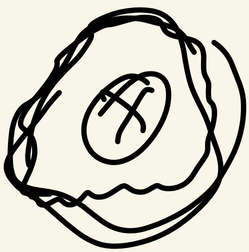
morphisms

~~invert~~ quasi-isomorphisms

$K_\bullet \xrightarrow{\sim} L_\bullet$

triangulated structure.

t-structure



$D(X)$

Stable ∞ category:

∞-categories contains
more info about
compatibilities, composition ...

$\rightsquigarrow$ info is intrinsic.

Where ∞-cats

- Alg geometry $\rightsquigarrow$ modern approach to motivic complexes $\rightsquigarrow$ stable ∞-category

intersection theory.

$\rightsquigarrow$ ~~derived~~ ∞-cat. of derived schemes.

$\sum \mathrm{Tor}(\,)\cdots$

$X \times_S Y$

$X \cdot Y$ $\rightsquigarrow$ intersection is always a derived scheme.

DGA I, II . . . XIII — ≤ HA

⤷ 80 page long

150

Grothendieck.

- Alg. Topo . . Fundamental ∞-groupoid.

. ∞ categ. of spectra ...

- Physics . quantum mechanics ~ ∞ .

QFT ....

. mechanics ~ symplectic geometry

multisymplectic geometry — fund. arg H

⤷ explained quite well in terms of

Lie-∞ Algebroid .

Now: precise maths.

chapter 1.1 and 2 of the Book.

- presheaves
- simplicial sets.

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presheaves.

A a category.

Def: A presheaf over A is

$$X : A^{op} \to Sets$$

$a \in ob(A)$, $X_a = X(a)$

$a \xrightarrow{u} b$, $X(u) = u^*$ X, Y.

$X, Y : A^{op} \longrightarrow Sets$

a natural transformation $X \longrightarrow Y$

$(f_a)\quad a \in A.$

$a, b \quad u : a \longrightarrow b$

$$\begin{tikzcd}
X_b \arrow[r, "u^*"] \arrow[d, "f_b"] \arrow[dr, phantom, "\circlearrowleft"] & X_a \arrow[d, "f_a"] \\
Y_b \arrow[r, "u^*"'] & Y_a
\end{tikzcd}
\qquad
X \overset{f}{\Longrightarrow} Y
\qquad
\begin{tikzcd}
A^{op} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow f"] & Sets
\end{tikzcd}$$

def <u>Yoneda embedding</u>

$$h : \begin{cases} A \longrightarrow \hat{A} \\ a \longmapsto hom_A(-, a) \end{cases}$$

presheaves over A.

Thm (Yoneda Lemma) $a \in A$, $X \in \hat{A}$

$$\mathrm{Hom}_{\hat{A}}(h_a, X) \xrightarrow[\text{natural}]{\sim} X_a$$

pf exo $\square$

Cor h is fully faithfull

pf) $$\mathrm{Hom}_{\hat{A}}(h_a, h_b) \overset{\text{Yoneda Lemma}}{\cong} h_b(a) = \mathrm{Hom}_A(a, b)$$

$$h(f) \longleftarrow f$$

$$\Downarrow$$

$$A \subseteq \hat{A} \qquad \left(\begin{array}{l} f : a \longrightarrow X \\ \Leftrightarrow h_a \xrightarrow{f} X \Leftrightarrow f \in X_a \end{array} \right)$$

Def $X \in \hat{A}$. Category of elements of X, A/X, $\int_A X$.

obj: (a, s), $a \in Ob(A)$,
$s \in X_a$
$a \xrightarrow{s} X$

morphisms: $(a, s) \longrightarrow (b, t)$

$u : a \longrightarrow b$

$u^* t = s$

$\shortparallel$

$X(u)(t)$.

$s : a \longrightarrow X$

$$\begin{tikzcd} h_a \arrow[rr, "u"] \arrow[dr, "s"'] & \arrow[d, phantom, "\circlearrowleft"] & h_b \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

$$\varphi_X : \begin{array}{ccc} A/X & \longrightarrow & \hat{A} \\ (a,s) & \longmapsto & h_a \\ u & \longmapsto & u \end{array}$$

faithful.

$$\cdots \longrightarrow (a,s) \xrightarrow{u} (b,t) \xrightarrow{v} (c,\delta) \cdots$$

$$\begin{tikzcd}
\cdots \arrow[r] & h_a \arrow[r, "u"] \arrow[dr, "s"'] & h_b \arrow[r, "v"] \arrow[d, "t"] & h_c \arrow[dl, "\delta"] \cdots \\
 & & X &
\end{tikzcd}$$

$\Rightarrow$ cocone on X.

Prop: $\boxed{X = \operatorname*{colim}_{(a,t) \in A/X} \varphi_X(a,t)}$

pf| $Y \in \hat{A}$

$b,t \qquad c,\delta$

$$\begin{tikzcd}
\cdots \arrow[r] & h_a \arrow[r, "u"] \arrow[dr, "f_{a,s}"] \arrow[ddr, "s"'] & h_b \arrow[r, "v"] \arrow[d, "f_{b,t}"] & h_c \arrow[r] \arrow[dl, "f_{c,\delta}"'] \arrow[ddl, "\delta"] & \cdots \\
& & Y \arrow[d, dashed, "\exists !"] & & \\
& & X & &
\end{tikzcd}$$

$f_{a,s} : h_a \to Y$

$\Longleftrightarrow f_s \in Y_a \quad \forall a, s$

~~Yoneda~~

$$(a,s) \xrightarrow{u} (b,t)$$

$$u^*(f_t) = f_s$$

$$\left\{ \begin{array}{l} X_a \to Y_a \\ s \mapsto f_s \end{array} \right\} \text{ morphism of presheaves} \qquad \square$$

Yoneda lemma for (locally small categories

$\mathrm{Hom}(A,B)$ are sets $\forall A, B$.

small category

ob a set

A is small $\Rightarrow \hat{A}$ locally small.

<u>Thm</u> (Kan) A small, $\mathcal{C}$ locally small category with small colimits.

$u : A \longrightarrow \mathcal{C}$ a functor.

$I \to \mathcal{C}$. small

evalu. at u

$$u^* : \begin{cases} \mathcal{C} \longrightarrow \hat{A} \\ X \longmapsto (a \mapsto \mathrm{Hom}_{\mathcal{C}}(u(a), X)) \end{cases}$$

has a ~~right~~ adjoint.

$u_! : \hat{A} \longrightarrow \mathcal{C}$. left i.e

$$\mathrm{Hom}_{\mathcal{C}}(u_! X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\hat{A}}(X, u^* C).$$

$X = h_a$ $\quad$ $\cancel{u(a)}$ $\quad$ $u(a) \cong u_! h_a$

$\mathrm{Hom}(h_a, u^* C) = \mathrm{Hom}(u_! h_a, C)$

$\quad\ \|$

$\mathrm{Hom}(u(a), C)$ $\quad$ Yoneda

$X \in \hat{A}$ $\quad$ $\alpha_X : A/X \longrightarrow \mathcal{C}$

$(a, s) \longmapsto u(a)$

$a \in A$, $\quad s \in X_a$

$$u_!(X) = \mathrm{colim}(\alpha_X).$$

$X = h_a$ $\quad$ $(b, t) \in A/h_a$

$$(b,t) \xrightarrow[s]{\exists!} (h_a, 1_a)$$

$$(b,t)\left(\begin{tikzcd} b \arrow[r, "t"] \arrow[dr, "t"'] & a \arrow[d, "1_a"] \\ & a \end{tikzcd}\right.$$

$$X = h_a, \quad U_!(h_a) = U(a).$$

$$\rightsquigarrow X \in \hat{A}, \quad Y \in \mathcal{C}.$$

$$\mathrm{Hom}_{\mathcal{C}}(U_! X, Y) = \mathrm{Hom}_{\mathcal{C}}\left(\operatorname*{colim}_{(a,s)} U(a), Y\right)$$

$$= \lim_{(a,s)} \underbrace{\mathrm{Hom}(U(a), Y)}_{U^*Y(a)}$$

$$\overset{\text{Yoneda}}{=} \lim_{(a,s)} \mathrm{Hom}_{\hat{A}}(h_a, U^*Y)$$

$$= \mathrm{Hom}_{\hat{A}}(\operatorname{colim} h_a, U^*Y)$$

$$\mathrm{Hom}_{\mathcal{C}}(u_! X, Y) \Rightarrow \mathrm{Hom}(X, u^* Y) \qquad \square$$

$u_!$ = extension of u by colimits.

one can show that $F : \hat{A} \to \mathcal{C}$ which preserves colimits.

$\exists\, u,\ F = u_!$ for.

$$F = \mathrm{id}_{\hat{A}} = h_!$$

Yoneda embedding.

<u>**Coro**</u>: $\hat{A} \xrightarrow{\text{functor}} \mathcal{C}$ commutes with colimits

$\Rightarrow$ it has a right adjoint.

$F : u_! \dashv u^*$. $\square$.

__Last remark__. $X \in \hat{A}$

study maps $Y \to X$

$\iff$ study $\widehat{A/X}$

why?:

$$(a,s) \mapsto a$$

~~Y/hA~~

$$\begin{tikzcd} A/X \arrow[r] \arrow[rr, bend right, "\nu"'] & A \arrow[r, "h"] & \hat{A} \end{tikzcd}$$

$\nu_!$ sends final object of A/X to X:

$$\widehat{A/X} \xrightarrow{\nu} \hat{A}/X$$

$$Y \to X$$

Simplicial sets

Def. Δ $\begin{cases} \text{objects} : [n] = \{0, \dots, n\}. \\ \text{maps} : \text{non-decreasing maps.} \end{cases}$

- A simplicial set is a presheaf $\Delta^{op} \xrightarrow{X} \text{Sets}$

$\forall n, \ X_n \in \text{Sets}$

sSets their category.

$\Delta^n = h_{[n]} :=$ standard n-simplex.

$\forall X \in \text{sSet}, \ X_n = X([n]) = \text{Hom}(\Delta^n, X)$

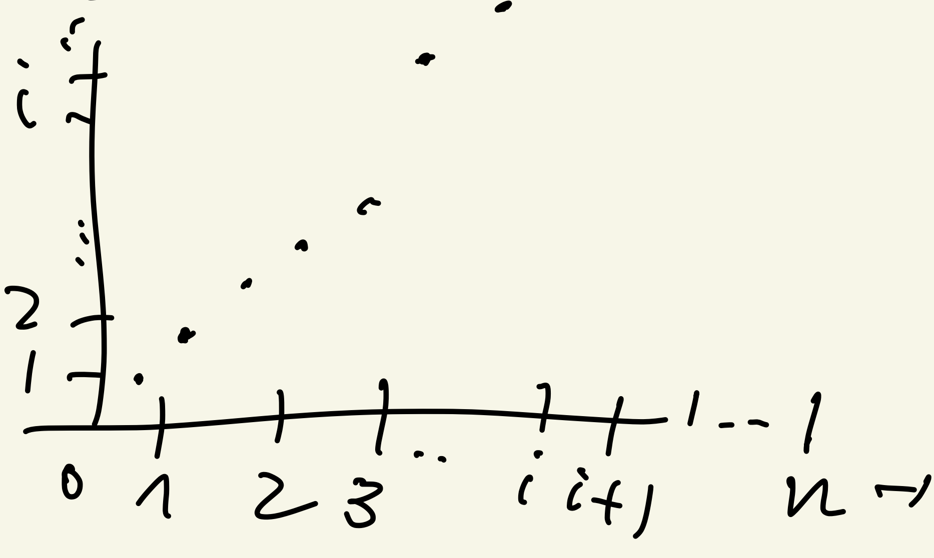
n-simplices of X, a simplex of X is a elt of on X_n.

giving a simplex $x \in X_n$

$\iff x : \Delta^n \longrightarrow X$.

$\partial_i^n : \left(\begin{array}{c} \Delta^{n-1} \longrightarrow \Delta^n \\ \cong \\ [n-1] \longrightarrow [n] \end{array} \right)$ face maps $\longrightarrow$

only map that misses i.

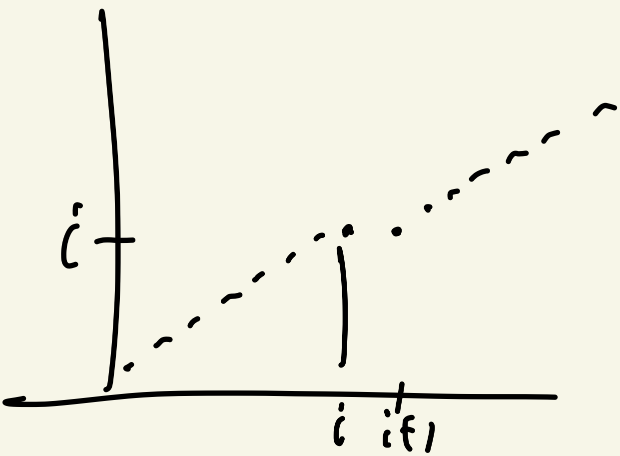


degeneracies maps.

$$\sigma_i^n : \Delta^{n+1} \longrightarrow \Delta^n \quad i \geq$$

$$\begin{tikzcd} \Delta^{n+1} \arrow[r] \arrow[d, phantom, "\wr\, g"] & \Delta^n \\ {[n+1]} \arrow[r] & {[n]} \end{tikzcd}$$

the only map that hits i twice.



identities

- $\partial_j^{n+1}\partial_i^n = \partial_i^{n+1}\partial_{j-1}^n \quad i<j$

- $\sigma_j^n\sigma_i^{n+1} = \sigma_i^n\sigma_{j+1}^{n+1} \quad i\leq j$

- $\sigma_j^{n-1}\partial_i^n = \begin{cases} \partial_i^{n-1}\sigma_{j-1}^{n-2} & i<j \\ id_{\Delta^{n-1}} & i=j,\ j+1 \\ \partial_{i-1}^{n-1}\sigma_j^{n-2} & i>j+1 \end{cases}$

no proof. $X \Leftrightarrow X_n,\ X_{n+1} \to X_n$

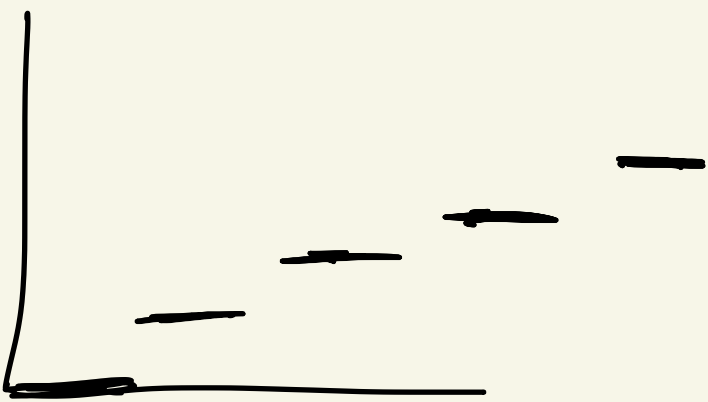
sset

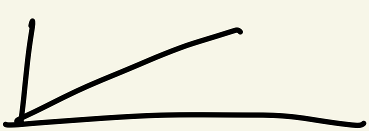
prop: $f : \Delta^m \longrightarrow \Delta^k \in \Delta$

has a unique factorisat° $f = i \circ c$

∂_i^n (i mono (strictly increasing map))

σ_j^n π split epi





Notat° $X \in sSet$

$$d_n^i = (\partial_i^n)^* : X_n \to X_{n-1}$$
$$s_n^i = (\sigma_i^n)^* : X_n \to X_{n+1}.$$

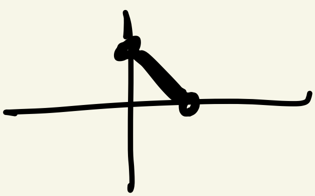
e.g. Topology.

type in the ball

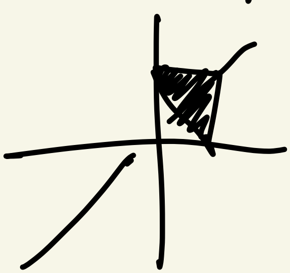
$$|\Delta^n| = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \;/\; \sum_{i=1}^n x_i \leq 1 \right\}$$

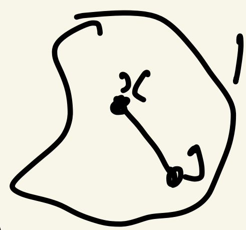
$n = 0$ $\cdot$

$n = 1$



$n = 2$





X.

— —

$f : [m] \longrightarrow [n]$ in Δ.

$|f| : [\Delta^m] \longrightarrow [\Delta^n]$

$|f|(x_0, \ldots x_m) = (y_0 \ldots y_n)$

$$y_j = \sum_{i \in f^{-1}(j)} x_i$$

$$\underbrace{\Delta}_{\text{small cat}} \xrightarrow{|\cdot|} \underbrace{\mathrm{Top}}_{\text{locally small with small colimits.}}$$

$\coprod \mathrm{Hom}/_\sim$

Kan $\Longrightarrow$ $\exists \left[\begin{array}{l} \mathrm{sSet} \longrightarrow \mathrm{Top} \\ X \longrightarrow |X| \end{array} \right]$

$\exists$ ~~[...]~~ r. adjoint : $\boxed{Y \longrightarrow \mathrm{Sing}\, Y}$.

$[n] \longrightarrow \mathrm{Ho}(|\Delta^n|, Y)$.

$$\mathrm{Hom}_{top}(|X|, Y) = \mathrm{Hom}_{sSets}(X, \mathrm{Sing}\,Y)$$

$\leadsto$ explains terminology $X \in sSet$

0 simplex $x : \Delta^0 \longrightarrow X$

$\Leftrightarrow$ point of X.

1-simplex path in X

$x = d_1^1(s) \longrightarrow y = d_1^0(s)$.

mms D.

$\boxed{\text{end}}$