Higher categories and homotopical algebra

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Inside Cisinski [1] :

3 - The homotopy theory of ∞-categories

3.1 - Kan fibrations and the Kan-Quillen model structure

Goal : construct the Kan-Quillen model category structure, encoding the homotopy theory of Kan complexes, and give a more precise description of this model structure.

1. Definitions for our structures
2. First properties
3. Bisimplicial sets
4. Partially ordered sets
5. Teasers

Summary

1. Definitions for our structures
   - Anodyne extensions
   - Kan fibrations
   - The Kan-Quillen model structure
2. First properties
3. Bisimplicial sets
4. Partially ordered sets
5. Teasers

Recall $\Delta^1 = Hom(-, [1])$.
$\Delta^1 \times (-)$ is an exact cylinder, ie an endofunctor of the category of simplicial sets, with $(\partial_1, \partial_2) : 1 \coprod 1 \to \Delta^1 \times (-)$ and $\sigma : \Delta^1 \times (-) \to 1$, satisfying some properties.

Definition 3.1.1.

An anodyne extension is an element of the smallest class of $\Delta^1 \times (-)$-anodyne maps.

See Example 2.4.13 for the construction of this smallest class, here for $I = \Delta^1 \times (-)$ and $S = \varnothing$:

1 - choose a cellular model M;

2 - set :

$$\Lambda_I^0(S, M) = S \cup \{I \otimes K \cup \{\epsilon\} \otimes L \to I \otimes L | K \to L \in M, \epsilon = 0, 1\};$$

3 - set by induction on n :

$$\Lambda_I^{n+1}(S, M) = \{I \otimes K \cup \partial I \otimes L \to I \otimes L | K \to L \in \Lambda_I^n(S, M)\};$$

4 - set :

$$\Lambda_I(S, M) = \cup_{n \geq 0} \Lambda_I^n(S, M);$$

5 - set :

$$An(S) = l(r(\Lambda_I(S, M)))$$

Proposition 3.1.2 (Gabriel and Zisman).

The following three classes of morphisms of simplicial sets are equal:
(a) the class of anodyne extensions;
(b) the smallest saturated class of maps containing inclusions of the form

$$\Delta^1 \times \partial\Delta^n \cup \{\epsilon\} \times \Delta^n \to \Delta^1 \times \Delta^n \quad \text{for } n \geq 0 \text{ and } \epsilon = 0, 1;$$

(c) the smallest saturated class containing inclusions of the form

$$\Lambda^n_k \to \Delta^n \quad \text{for } n \geq 1 \text{ and } 0 \leq k \leq n.$$

Proof of $(a) = (b)$.

To build $An(S)$ using our algorithm, take $M = \{\partial\Delta^n \to \Delta^n,\ n \geq 0\}$.
Goal : prove $l(r(\Lambda_l(S,M))) = l(r(\Lambda_l^0(S,M)))$. We already have the the reverse inclusion because $\Lambda_l(S,M) \supset \Lambda_l^0(S,M)$.
Let's show $\Lambda_l(S,M) \subset l(r(\Lambda_l^0(S,M)))$. Actually, sufficient to prove that for any monomorphism $K \to L$ and $\epsilon = 0, 1$, the inclusion :

$$\Delta^1 \times K \cup \{\epsilon\} \times L \to \Delta^1 \times L$$

belongs to $l(r(\Lambda_l^0(S,M)))$. This is true for all the maps of M. And M is a cellular model, and the above property is stable under saturation, so it is true for all monomorphisms.

Lemma 3.1.3.

The following two classes of morphisms are equal:

(a) the smallest saturated class of maps containing inclusions of the form

$$\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \to \Delta^1 \times \Delta^n \quad \text{for } n \geq 0 \text{ and } \epsilon = 0, 1;$$

(b) the smallest saturated class containing inclusions of the form

$$\Lambda^n_k \to \Delta^n \quad \text{for } n \geq 1 \text{ and } 0 < k \leq n.$$

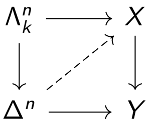
Corollary 3.1.4.

For any anodyne extension $K \to L$ and any monomorphism $U \to V$, the induced inclusion $K \times V \cup L \times U \to L \times V$ is an anodyne extension.

Definition 3.1.5.

A Kan fibration is a morphism of simplicial sets with the right lifting property with respect to the inclusions of the form $\Lambda^n_k \to \Delta^n$, for $n \geq 1$ and $0 \leq k \leq n$.
A Kan complex is a simplicial set X such that the morphism from X to the final simplicial set is a Kan fibration.

Ie :



Corollary 3.1.6.

For any monomorphism $i : U \to V$ and for any Kan fibration $p : X \to Y$, the canonical map

$$(i^*, p_*) : \underline{Hom}(V, X) \to \underline{Hom}(U, X) \times_{\underline{Hom}(U,Y)} \underline{Hom}(V, Y)$$

is a Kan fibration.

Corollary 3.1.7.

For any anodyne extension $i : K \to L$ and for any Kan fibration $p : X \to Y$, the canonical map

$$(i^*, p_*) : \underline{Hom}(L, X) \to \underline{Hom}(K, X) \times_{\underline{Hom}(L,Y)} \underline{Hom}(K, Y)$$

is a trivial fibration.

Theorem 3.1.8.

There is a unique model category structure on the category of simplicial sets whose class of cofibrations is the class of monomorphisms, and whose fibrant objects are the Kan complexes. Moreover, any anodyne extension is a trivial cofibration, and the fibrations between fibrant objects exactly are the Kan fibrations between Kan complexes.

Definition 3.1.9

The structure we get will be called the Kan-Quillen model category structure.
Its weak equivalences will be called the weak homotopy equivalences.

Proof.

The theorem 2.4.19, applied to the category of simplicial sets for $\Delta^1 \times (-)$ and for An, gives us the model category structure, where :

- the cofibrations are the monomorphisms;
- weak equivalences are the morphisms $f : X \to Y$ such that, for any fibrant presheaf W, $f^* : [Y, W] \to [X, W]$ is bijective;
- naive fibrations are the morphisms having the right lifting property with respect to An;
- the fibrant presheaves X are the ones such that the map from X to the final preashef is a naive fibration.

Then, use proposition 3.1.2 to rewrite An as the smallest saturated class containing the $\Lambda^n_k \to \Delta^n$.

Summary

1 Definitions for our structures

2 First properties
- Useful lemmas
- An extension property

3 Bisimplicial sets

4 Partially ordered sets

5 Teasers

Corollary 3.1.10.

The class of weak homotopy equivalences is closed under finite products.

Proof.

Equivalent to prove that for any simplicial set Y, the functor $X \mapsto X \times Y$ preserves weak equivalences. By corollary 3.1.4 it preserves anodyne extensions. Then use proposition 2.4.40 to show it preserves trivial cofibrations. As it also preserves cofibrations, then it is part of a Quillen adjunction. And as all simplicial sets are cofibrant, then it will preserve weak equivalences.

Lemma 3.1.11.

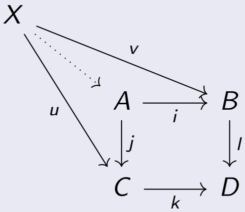
Let $\mathcal{C}$ be a category in which a commutative square

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle l} \\ C & \xrightarrow[k]{} & D \end{array}$$

is given. Assume that the maps j, k, l have retractions r, q, p, respectively, and that $pk = ir$. Then this square is absolutely Cartesian.

Proof.

It is sufficient to check that the square is Cartesian. If $u : X \to C, v : X \to B$ are such that $ku = lv$, then taking $w = ru : X \to A$ leads to $kjw = ku$ and $liw = lv$ (using the retractions), but k, l are monomorphisms, thus the existence of w. For the unicity, use the retraction of j : then $w = rjw = ru$.



Lemma 3.1.11.

Let $n \geq 2$ and $0 \leq i < j \leq n$, the following square is absolutely Cartesian :

$$\begin{array}{ccc} \Delta^{n-2} & \xrightarrow{\delta_i^{n-1}} & \Delta^{n-1} \\ {\scriptstyle \delta_{j-1}^{n-1}}\downarrow & & \downarrow{\scriptstyle \delta_j^n} \\ \Delta^{n-1} & \xrightarrow[\delta_i^n]{} & \Delta^n \end{array}$$

Recall (theorem 1.1.10) : extension by colimits :
Given a small category D and a locally small category $\mathcal{C}$ which has small colimits, any functor $u : D \to \mathcal{C}$ can be 'extended' to a functor $u_! : \hat{D} \to \mathcal{C}$, where $u_!$ is the left adjoint of the functor of evaluation at u.

Proposition 3.1.13.

Let $\mathcal{C}$ be a small category and $A : \Delta \to \hat{\mathcal{C}}$ a functor. Then the induced colimit preserving functor $A_! : sSet \to \hat{\mathcal{C}}$ preserves monomorphisms if and only if the map

$$(A(\delta_0^1), A(\delta_1^1)) : A^0 \coprod A^0 \to A^1 = A_!(\Delta^1)$$

is a monomorphism.

Sketch of proof.

Recall that the class of monomorphisms of simplicial sets is the smallest saturated class containing the inclusions $\partial\Delta^n \to \Delta^n$. So it suffices to checks these inclusions are sent to monomorphisms. For $n = 0$: by cocontinuity of $A_!$. And for $n \geq 2$, they are always sent to a monomorphism (using last lemma).

Proposition 3.1.14.

Let $A, B : \Delta \to \mathcal{C}$ be two functors with values in a model category. We denote by $A_!$ and $B_!$ their extensions by colimits, respectively, and we assume that both of them send monomorphisms to cofibrations. If a natural transformation $u : A \to B$ induces a weak equivalence $A^n \to B^n$ for all $n \geq 0$, then, for any simplicial set X, the map

$$u_X : A_!(X) \to B_!(X)$$

is a weak equivalence.

Proof.

Consider the class of X such that u_X is a weak equivalence. By corollaries 2.3.16, 2.3.16 and 2.3.29, it is saturated by monomorphisms (ie stable under coproducts, pushouts, countable limits). Moreover it contains all representable presheaves so we can apply corollary 1.3.10.

Summary

1. Definitions for our structures
2. First properties
3. Bisimplicial sets
   - Definition
   - Diagonal
4. Partially ordered sets
5. Teasers

Definition

A bisimplicial set is a presheaf over the product $\Delta \times \Delta$.

- For a bisimplicial set X, we write $X([m],[n]) = X_{m,n}$.
- For simplicial sets Y, Z, we set $(Y \boxtimes Z)_{m,n} = Y_m \times Z_n$.
- For a bisimplicial set X and a simplicial set K, write X^K for the simplicial set defined by :

$$(X^K)_m = \varprojlim_{\Delta^n \to K} X_{m,n}$$

  So $K \mapsto X^K$ is the extension by colimits of the functor $\Delta \to sSet$ sending $[n]$ to $X^{\Delta^n} = ([m] \mapsto X_{m,n})$.
- For a bisimplicial set X, write $diag(X)$ for the simplicial set defined by :

$$diag(X)_n = X_{n,n}$$

Theorem 3.1.16.

If a morphism of bisimplicial sets $X \to Y$ is a levelwise weak homotopy equivalence (i.e. induces a weak homotopy equivalence $X^{\Delta^m} \to Y^{\Delta^m}$ for all $m \geq 0$), then the diagonal map

$$diag(X) \to diag(Y)$$

is a weak homotopy equivalence.

Sketch of proof.

Main point : use proposition 3.1.14 and the fact that $K \mapsto X^K$ is the extension by colimits of $[n] \mapsto X^{\Delta^n}$ to conclude that $X^K \to Y^K$ is a weak equivalence for any simplicial set K.

Summary

1. Definitions for our structures
2. First properties
3. Bisimplicial sets
4. Partially ordered sets
    - Definitions : Sd, Ex, a, b
    - Two weak homotopy equivalences
5. Teasers

Definition

- Given a partially ordered set E, write $s(E)$ for the set of finite non-empty totally ordered subsets of E, ordered by inclusion.
- We get a functor s from the category of partially ordered sets to the category of small categories.
- Set $Sd : sSet \to sSet$ as the extension by colimits of $[n] \mapsto s([n])$.
- Its right adjoint Ex is $Ex(X)_n = Hom(Sd(\Delta^n), X)$.

Definition

- $a_n : s([n]) \to [n], U \mapsto \max(U)$ induces by extension by colimits the natural transformation $a_X : Sd(X) \to X$.
- By transposition we get $b_X : X \to Ex(X)$, obtained by composing the Yoneda isomorphism $X_n \simeq Hom(\Delta^n, X)$ with the map $a_n^* : Hom(\Delta^n, X) \to Hom(Sd(\Delta^n), X)$.

Proposition 3.1.18.

The functor Sd preserves monomorphisms as well as anodyne extensions.

Proposition 3.1.19

For any simplicial set X, a_X is a weak homotopy equivalence.

Proof.

By proposition 3.1.14, it suffices to prove that a_X is a weak homotopy equivalence for $X = \Delta^n$, $n \geq 0$. Use the two-out-of-three property for weak homotopy equivalences to conclude.

Lemma 3.1.20

Let $f, g : K \to L$ be two morphisms of simplicial sets, such that there exists a homotopy $h : \Delta^1 \times K \to L$ from f to g. Then, for any simplicial set X, there exists a homotopy from f^* to g^* (precompositions).

Proof.

h induces :

$$h^* : Hom(L, X) \to Hom(\Delta^1 \times K, X) \simeq Hom(\Delta^1, Hom(K, X))$$

which induces by transposition :

$$\tilde{h} : \Delta^1 \times Hom(L, X) \to Hom(K, X)$$

Proposition 3.1.21

For any simplicial set X, b_X is a weak homotopy equivalence.

Summary

1 Definitions for our structures

2 First properties

3 Bisimplicial sets

4 Partially ordered sets

5 Teasers
- The functor Ex^{∞}
- The functor π_0

Definition.

For a simplicial set X, we set :

$$Ex^{n+1} = Ex(Ex^n(X))$$

b induces a sequence :

$$X \to Ex(X) \to Ex^2(X) \to \dots$$

Define $Ex^\infty(X)$ as its limit.

Theorem 3.1.29 (Quillen).

A morphism of simplicial sets is a fibration of the Kan-Quillen model category structure if and only if it is a Kan fibration. A morphism of simplicial sets is a trivial cofibration of the Kan-Quillen model category structure if and only if it is an anodyne extension.

Definition.

Define π_0 as the left adjoint of the inclusion functor $Set \to sSet$.

Proposition 3.1.31.

The functor π_0 sends weak homotopy equivalences to bijections and commutes with finite products. Furthermore, for any Kan complex X, the set $\pi_0(X)$ may be identified with the set of Δ^1-homotopy classes of maps $\Delta^0 \to X$.

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