Étale motives of geometric origin

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Abstract

Over qcqs finite-dimensional schemes, we prove that étale motives of geometric origin can be characterised by a constructibility property which is purely categorical, giving a full answer to the question "Do all constructible étale motives come from geometry?" which dates back to Cisinski and Déglise's work. We also show that they afford the continuity property and satisfy h-descent and Milnor excision.

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1 Introduction.

Consider a scheme X and a commutative ring of coefficients Λ. We denote by $\mathrm{DM}^{\text{ét}}(X,\Lambda)$ the stable ∞-category of étale motives defined as the Tate-stabilisation of the category of $\mathbb{A}^1$-local étale hypersheaves of Λ-modules over the category of smooth S-schemes. This version of étale motives without transferts was studied by Ayoub in [Ayo14]; it is equivalent over noetherian schemes to the category of h-motives studied by Cisinski and Déglise in [CD16]. It is endowed with the six functors formalism by [AGV22, Theorem 4.6.1]. Denote by Λ_X the unit object of $\mathrm{DM}^{\text{ét}}(X,\Lambda)$.

In this paper we explore some properties of étale motives of geometric origin:

Definition. Let X be a scheme and let Λ be a commutative ring. Let M be an object of $\mathrm{DM}^{\text{ét}}(X,\Lambda)$. We say that the motive M is *of geometric origin* (or simply *geometric*) if it belongs to the thick subcategory $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X,\Lambda)$ of $\mathrm{DM}^{\text{ét}}(X,\Lambda)$ generated by the $f_\sharp(\Lambda_Y)(n)$ for $f\colon Y\to X$ smooth.

Our main goal is to prove that motives of geometric origin form an étale hypersheaf and to give a purely categorical characterisation of geometric étale motives:

Main Theorem. *(Theorem 4.1 and Lemma 2.4) Let Λ be a commutative ring.*

1. *Let X be a qcqs finite-dimensional scheme. An étale motive M over X is of geometric origin if and only if it is* constructible*, that is for any open affine $U\subseteq X$, there is a finite stratification $U_i\subseteq U$ made of constructible locally closed subschemes such that each $M|_{U_i}$ is dualisable.*

2020 *Mathematics subject classification.* 14F42, 14F08
Key words and phrases. Voevodsky motives, constructible sheaves, descent.

2. *The functor* $\mathrm{DM}^{\text{ét}}_{\text{gm}}(-,\Lambda)$ *is an étale hypersheaf over qcqs finite-dimensional schemes.*

This question dates back to [CD16], where Cisinski and Déglise considered étale-locally geometric[1] motives and showed that over noetherian schemes, the latter coincides with constructible motives. To prove the above theorem, we need to generalise to non-noetherian schemes the continuity properties proved in *loc. cit.*

Theorem. *(Continuity; Theorem 3.2) Let* Λ *be a commutative ring. Consider a qcqs finite-dimensional scheme* X *that is the limit of a projective system of qcqs finite-dimensional schemes* (X_i) *with affine transition maps. Then, the natural map*

$$\operatorname{colim} \mathrm{DM}^{\text{ét}}_{\text{gm}}(X_i,\Lambda) \to \mathrm{DM}^{\text{ét}}_{\text{gm}}(X,\Lambda)$$

is an equivalence.

Using continuity, the most critical case of the main theorem is the case where $X = \mathrm{Spec}\ (k)$ with k a field and where Λ is regular and flat over $\mathbb{Z}$. In that particular case, the result can be deduced from the case of rational coefficients which was proved in [CD16] and results of [Rui22] on Artin motives. The latter are consequences of results of [BG23] which imply that étale sheaves of the form $f_\sharp\Lambda_L$ for L/k a finite separable extension, generate the category of étale-locally perfect constant sheaves as a thick subcategory of itself.

We end the paper with some non-trivial descent properties of geometric étale motives. Namely we prove that they satisfy h-descent in the sense of [Stacks, Tag 0ETQ] generalizing [CD16, Corollary 5.5.7.] to the non-noetherian setting and we prove that they satisfy Milnor excision in the sense of [EHIK21a], giving a generalisation of the analogous of [EHIK21b, Theorem 5.6] for geometric motives:

Theorem. *(Theorem 5.2 and Proposition 5.3) Let* Λ *be a commutative ring. The functor* $\mathrm{DM}^{\text{ét}}_{\text{gm}}(-,\Lambda)$ *satisfies Milnor excision and h-descent over qcqs finite-dimensional schemes.*

Remark. In an upcoming paper, we will define a category of geometric Nori motives with coefficients in an arbitrary commutative ring Λ over qcqs finite-dimensional schemes and show that it satisfies arc-descent. We therefore expect $\mathrm{DM}^{\text{ét}}_{\text{gm}}(-,\Lambda)$ to satisfy arc-descent.

2 First descent properties.

Until the end of the proof of the main theorem, we will use the following notation:

Definition 2.1. Let X be a scheme and let Λ be a commutative ring. We denote by $\mathrm{DM}^{\text{ét}}_{c}(X,\Lambda)$ the subcategory of $\mathrm{DM}^{\text{ét}}(X,\Lambda)$ made of *constructible motives*, that is those motives such that for any open affine $U \subseteq X$, there is a finite stratification $U_i \subseteq U$ made of constructible locally closed subschemes such that each $M|_{U_i}$ is dualisable.

Remark 2.2. The functors of type f^*, $f_\sharp$ and $\otimes$ readily preserve geometric objects. The functor i_*, for i a closed immersion, preserves geometric objects by using the localisation triangle.

On the other hand, the functors of type f^* and $\otimes$ preserve constructible objects. The functors $j_\sharp$ and i_*, for j an open immersion and i a closed immersion, also preserve constructible objects.

[1] Beware that in *loc. cit.*, they call *constructible* what we chose to call *of geometric origin* in this paper. We chose this terminology because they are the motives that can be built from objects coming from geometry. In addition, objects that are called *locally constructible* by Cisinski and Déglise are more similar to classical constructible sheaves. In any case, the purpose of this paper is to show that all those notions coincide.

We first prove descent properties for geometric objects and for constructible objects.

Lemma 2.3. *Let Λ be a commutative ring. The functor $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\Lambda)$ is a Nisnevich hypersheaf over qcqs schemes.*

Proof. It suffices to show that $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\Lambda)$ it is a subsheaf of $\mathrm{DM}^{\text{ét}}(-,\Lambda)$ (a subsheaf of a hypersheaf is itself a hypersheaf for qcqs sites). Assume that (U,V) is a Nisnevich cover of X with U open and $V\to X$ étale and let M be a motive over X. Then if $M|_U$ and $M|_V$ are geometric, so is $M|_{U\times_X V}$, so that by Nisnevich excision, the motive M itself is geometric. □

Lemma 2.4. *Let Λ be a commutative ring. The functor $\mathrm{DM}^{\text{ét}}_{c}(-,\Lambda)$ is an étale hypersheaf over qcqs schemes.*

Proof. It suffices to show that $\mathrm{DM}^{\text{ét}}_{c}(-,\Lambda)$ is a subsheaf of $\mathrm{DM}^{\text{ét}}(-,\Lambda)$. The proof is very similar to the proof of [HRS23, Lemma 4.5], so we only give a sketch.

Take a qcqs scheme X and an étale-locally constructible motive M over X. We can assume X to be affine and to have a surjective étale map $f\colon Y\to X$ such that $f^*(M)$ is constructible. We then reduce to f being finite étale using [Stacks, Tag 03S0] and then to the case where f is a torsor under some finite group G: writing X as a limit of noetherian schemes, the map f is the pullback of some étale map defined over a noetherian scheme which we can then replace by a torsor under some finite group.

Finally, we can construct a G-equivariant stratification of Y from the stratification $\mathcal{Y}$ that witnesses the constructibility of $f^*(M)$. This yields a stratification of X whose pullback to Y is $\mathcal{Y}$. Hence, replacing X with one of its strata, we can assume that $f^*(M)$ is dualisable; as being dualisable is étale-local, the motive M is also dualisable which finishes the proof. □

3 Continuity.

We now prove the continuity property for geometric motives. We first need some compatibilities with rationalisation:

Proposition 3.1. *Let Λ be a commutative ring, let $f\colon Y\to X$ be a morphism of qcqs finite-dimensional schemes, let M and N in $\mathrm{DM}^{\text{ét}}(X,\Lambda)$ be motives over X with M constructible or geometric and let P be a motive over Y. Then, the natural transformations below are equivalences:*

1. $\mathrm{Map}(M,N)\otimes\mathbb{Q}\to\mathrm{Map}(M,N\otimes\mathbb{Q})$.
2. $\underline{\mathrm{Hom}}(M,N)\otimes\mathbb{Q}\to\underline{\mathrm{Hom}}(M,N\otimes\mathbb{Q})$.
3. $f_*(P)\otimes\mathbb{Q}\to f_*(P\otimes\mathbb{Q})$.
4. $i^!(N)\otimes\mathbb{Q}\to i^!(N\otimes\mathbb{Q})$ *(for i a closed immersion).*

Proof. We begin with the case where M is geometric. In this proof, we say that M is rationalisable if

$$\mathrm{Map}(M,N)\otimes\mathbb{Q}\to\mathrm{Map}(M,N\otimes\mathbb{Q})$$

is an equivalence. Rationalisable objects are closed under the functors of type $g_\sharp$. Therefore, to prove (1), we can assume that $M=\Lambda_X$. Note first that if K is a complex of étale sheaves of abelian groups, the canonical map:

$$\mathrm{H}^i_{\text{ét}}(X,K)\otimes\mathbb{Q}\to\mathrm{H}^i_{\text{ét}}(X,K\otimes\mathbb{Q})$$

is an isomorphism for any integer i. Indeed, [CM21, Corollary 3.29] gives a bound to the étale cohomological dimension with rational coefficients over qcqs finite-dimensional schemes, and the above assertion then follows from [CD16, Lemma 1.1.10]. By the same argument as [CD16, Corollary 5.4.8] this gives (1).

Geometric objects generate $\mathrm{DM}^{\text{ét}}(X,\Lambda)$ as a localizing subcategory of itself. Hence to show that the maps of (2), (3) and (4) are equivalences, it suffices to show that they become equivalences after applying $\mathrm{Map}(C,-)$ for C geometric. Since the functors of type f^*, i_* and $\otimes$ preserve geometric objects, this follows by adjunction from the equivalence of (1).

We now tackle the case where M is constructible. The third equivalence implies that the functors of type g^* preserve rationalisable objects. Hence, being rationalisable is Zariski-local: if we have a finite Zariski cover $\{U_i \to X\}$ and a motive M over X such that the $M|_{U_i}$ are rationalisable, so are the restrictions of M over the intersections of the U_i; as we can write M as a finite colimit with terms of the form $(V \to X)_\sharp(M|_V)$ with V an intersection of the U_i, the motive M is also rationalisable. Hence, we can assume X to be affine.

Note now that (4) implies that rationalisable objects are preserved by the functor i_* for i a closed immersion. By dévissage and localisation, we are therefore reduced to the case where M is dualisable. In that case, replacing N by $N \otimes M^\vee$ we can further reduce to the case where $M = \Lambda_X$ which we already proved.

Assertion (2) in the constructible setting can be proved in the same way as in the geometric setting. □

Theorem 3.2. *(Continuity) Let Λ be a commutative ring. Consider a qcqs finite-dimensional scheme X that is the limit of a projective system of qcqs finite-dimensional schemes (X_i) with affine transition maps. Then, the natural map*

$$\mathrm{colim}\, \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X_i,\Lambda) \to \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\Lambda)$$

is an equivalence.

In the course of the proof, we will need the following lemma.

Lemma 3.3. *Let Λ be a commutative ring. Assume that Theorem 3.2 holds with coefficients Λ. Then, geometric motives with coefficients Λ are constructible.*

Proof. Let X a qcqs scheme. Write X as a limit of noetherian finite-dimensional schemes X_i with affine transition maps. Any geometric motive over X_i is constructible using [CD16, Theorem 6.3.26]. If M is a geometric object, it is the pullback of some M_i over some X_i because we assumed Theorem 3.2 to be true. As the motive M_i is constructible, so is M. □

Proof. (of Theorem 3.2) Once full faithfulness is proven, the essential surjectivity is easy: we first reduce by Zariski descent to objects of the form $f_\sharp(\Lambda_Y)(n)$ with Y affine and $f\colon Y \to X$ smooth. The usual continuity properties of [Gro66, Théorème 8.10.5, Proposition 17.7.8] then give the result. Therefore we tackle the full faithfulness.

Let (M_i) and (N_i) be cartesian sections over the diagram of schemes (X_i) of the functor $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\Lambda)$, and let M and N be their respective pullbacks along the maps $X \to X_i$, we want to prove that the canonical map

$$\mathrm{colim}\, \mathrm{Map}_{\mathrm{DM}^{\text{ét}}}(M_i,N_i) \to \mathrm{Map}_{\mathrm{DM}^{\text{ét}}}(M,N)$$

is an equivalence.

First assume that Λ is flat over the integers (so that $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is a non-derived ring for all prime p). The above equivalence be tested after tensoring with $\mathbb{Q}$ and $\mathbb{Z}/p\mathbb{Z}$ for any prime p. Tensoring with $\mathbb{Z}/p\mathbb{Z}$ is a finite limit and therefore gets inside mapping spectra. Using Proposition 3.1 this is also the case for tensoring with $\mathbb{Q}$, thus we can assume that either Λ is a $\mathbb{Q}$-algebra, or that Λ is a $\mathbb{Z}/p\mathbb{Z}$-algebra. The case of $\mathbb{Q}$-algebras is already known by [EHIK21b, Lemma 2.2 (i), Lemma 5.1 (ii)]. Using the classical argument with the Artin-Schreier exact sequence given in [CD16, Proposition A.3.4], we can assume p to be invertible in X and then use the rigidity theorem [BH21, Theorem 3.1], so that it suffices to prove the statement for constructible étale sheaves with torsion coefficients which was proved in [HS23, Theorem 2.2].

Assume now that we know the result for a ring R, we show that the result still holds for $\Lambda = R/(f)$ for any element f of R. Let 0 be an element of our index set, so that M_i (resp. N_i) is the pullback to X_i of M_0 (resp. N_0) for $i \geqslant 0$. To prove the result, by dévissage, we may assume that M_0 and N_0 are generators. In particular, they are isomorphic to $M_0'/(f)$ and $N_0'/(f)$ for some generators M_0' and N_0' of $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X_0, R)$. But for those the mapping spectra is the cofiber of the multiplication by f, so that we obtain the result.

Hence, the result is true when $\Lambda = A/J$, where A is flat over $\mathbb{Z}$ and J is a finitely generated ideal of A. In particular, the result is true for any ring of finite type over $\mathbb{Z}$.

In general, we may write $\Lambda = \mathrm{colim}_i \Lambda_i$ as a filtered colimit of rings of finite type over the integers and we have the following Lemma 3.4.

Lemma 3.4. *Let $\Lambda = \mathrm{colim}_{j\in J} \Lambda_j$ be a colimit of commutative rings and let X be a qcqs scheme. Then the natural functor*

$$\mathrm{colim}_j\, \mathrm{DM}^{\text{ét}}_{\text{gm}}(X, \Lambda_j) \to \mathrm{DM}^{\text{ét}}_{\text{gm}}(X, \Lambda)$$

is an equivalence.

Proof. This proof was communicated to the authors by Denis-Charles Cisinski. First, we prove that the functor is fully faithful. Let (M_j) and (N_j) be objects of the colimit, and let 0 be an element of the index set, so that M_j and N_j are defined over Λ_0 for $j \geqslant 0$. By dévissage we may assume that M_0 and N_0 are generators, so that they are in fact of the form $M \otimes_{\mathbb{Z}} \Lambda_0$ and $N \otimes_{\mathbb{Z}} \Lambda_0$ with M and N in $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X, \mathbb{Z})$. Thus (using the $\otimes_{\mathbb{Z}}\Lambda$ adjunctions) we want to prove that the map

$$\mathrm{colim}_j\, \mathrm{Map}_{\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})}(M, N \otimes_{\mathbb{Z}} \Lambda_j) \to \mathrm{Map}_{\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})}(M, N \otimes_{\mathbb{Z}} \Lambda)$$

is an equivalence. This can be checked after tensoring with $\mathbb{Q}$ and with $\mathbb{Z}/p\mathbb{Z}$ for all primes p. The map, when tensored by $\mathbb{Q}$ is an equivalence because by Proposition 3.1 it gives the analogous map with the integers replaced by $\mathbb{Q}$, and the latter is an equivalence because $M \otimes \mathbb{Q}$ is then a compact object of $\mathrm{DM}^{\text{ét}}(X, \mathbb{Q})$ by [EHIK21b, Lemma 5.1(i)]. After tensorisation by $\mathbb{Z}/p\mathbb{Z}$ (which is a finite colimit, hence commutes with the colimit and the Map), we obtain by rigidity the analogous map in $\mathrm{D}(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$:

$$\mathrm{colim}_j\, \mathrm{Map}_{\mathrm{D}(X_{\text{ét}},\mathbb{Z}/p\mathbb{Z})}(M\otimes_{\mathbb{Z}}\mathbb{Z}/p\mathbb{Z}, N\otimes_{\mathbb{Z}}\Lambda_j\otimes_{\mathbb{Z}}\mathbb{Z}/p\mathbb{Z}) \to \mathrm{Map}_{\mathrm{D}(X_{\text{ét}},\mathbb{Z}/p\mathbb{Z})}(M\otimes_{\mathbb{Z}}\mathbb{Z}/p\mathbb{Z}, N\otimes_{\mathbb{Z}}\Lambda\otimes_{\mathbb{Z}}\mathbb{Z}/p\mathbb{Z}).$$

Now, each complex $\Lambda_j \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is concentrated in degrees $[-1, 0]$ hence we are looking at a colimit which is uniformly bounded, and $M \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is constructible by Lemma 3.3 applied to $\Lambda = \mathbb{Z}$. As filtered colimits of uniformly bounded objects commute with mapping spectra from constructible complexes by [HRS23, Lemma 5.3], the full faithfulness holds. The essential surjectivity is easy, as it is clear that generators of $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X, \Lambda)$ are in the image. ∎

Finally, we have that

$$\mathrm{colim}_i\, \mathrm{DM}^{\text{ét}}_{\text{gm}}(X_i,\Lambda) \simeq \mathrm{colim}_i \mathrm{colim}_j\, \mathrm{DM}^{\text{ét}}_{\text{gm}}(X_i,\Lambda_j) \simeq \mathrm{colim}_j\, \mathrm{DM}^{\text{ét}}(X,\Lambda_j) \simeq \mathrm{DM}^{\text{ét}}_{\text{gm}}(X,\Lambda),$$

finishing the proof of the theorem. ☐

Corollary 3.5. *Let Λ be a commutative ring and let X be a finite-dimensional qcqs scheme.*

1. *Let $f\colon Y \to X$ be a separated finite type morphism. Then the functor $f_!$ restricts to a functor*

$$f_!\colon \mathrm{DM}^{\text{ét}}_{\text{gm}}(Y,\Lambda) \to \mathrm{DM}^{\text{ét}}_{\text{gm}}(X,\Lambda)$$

   between geometric motives.

2. *The ∞-category $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X,\Lambda)$ is the thick subcategory of $\mathrm{DM}^{\text{ét}}(X,\Lambda)$ generated by the $p_*(\Lambda_Y)(n)$ for $p\colon Y \to X$ projective and $n \in \mathbb{Z}$.*

3. *The ∞-category $\mathrm{DM}^{\text{ét}}(X,\Lambda)$ is the localising subcategory of itself generated by the $p_*(\Lambda_Y)(n)$ for $p\colon Y \to X$ projective and $n \in \mathbb{Z}$.*

Proof. For the first point, assume first that $f = p$ is proper. Let M be an object of $\mathrm{DM}^{\text{ét}}_{\text{gm}}(Y,\Lambda)$, and write $X = \lim X_i$ as a limit of schemes that are of finite type over the integers. By [Gro66, Théorème 8.10.5.(xii)] one can find a $p_{i_0}\colon Y_{i_0} \to X_{i_0}$ proper such that the pullback to X is p. In particular

$$Y = \lim_{i \geqslant i_0} Y_{i_0} \times_{X_{i_0}} X_i$$

and there is some i such that M is the pull back of a motive N over Y_i by continuity. Then by [CD19, Proposition 4.2.11] the object $(p_i)_*M_i$ is geometric in $\mathrm{DM}^{\text{ét}}(X_i,\Lambda)$. Thus, its pullback to X is geometric, but by the proper base change theorem this is p_*M. Finally, if f is separated of finite type, choosing a Nagata compactification of f yields the result.

Now for the second point, write $X = \lim_i X_i$ as a limit of schemes that are of finite type over the integers. Let M be an object of $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X,\Lambda)$, that comes by continuity above from an object M_i of $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X_i,\Lambda)$ for some index i. Thanks to [Ayo07, Lemme 2.2.23] we know that the result is true for $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X_i,\Lambda)$, so that there exists a finite diagram

$$F\colon J \to \mathrm{DM}^{\text{ét}}_{\text{gm}}(X_i,\Lambda)$$

such that $F(j)$ is of the form $p^j_*\Lambda(n_j)$ for p^j projective and $n_j \in \mathbb{Z}$, and such that M_i is a direct factor of $\mathrm{colim}_J F$. Thus, the object M is a direct factor of $(\mathrm{colim}_J F)_{|X}$, which by proper base change is the colimit of the objects $q^j_*\Lambda(n_j)$, with q^j the base change of p^j to X. This finishes the proof, as the third point readily follows from the second one. ☐

Combining Theorem 3.2 and Lemma 3.3, we proved:

Corollary 3.6. *Geometric motives are constructible.*

We now prove the converse when X is Λ-finite by linking both notions to compactness:

Proposition 3.7. *Let Λ be a commutative ring. Let X be a qcqs Λ-finite scheme and let M be an object of $\mathrm{DM}^{\text{ét}}(X,\Lambda)$. Then, the following are equivalent:*

1. *M is of geometric origin.*

2. *M is constructible.*

3. *The functor $\underline{\mathrm{Hom}}(M,-)$ is compatible with colimits.*

4. *M is a compact object of $\mathrm{DM}^{\text{ét}}(X,\Lambda)$.*

Proof. The equivalence between geometric objects and compact objects is [EHIK21b, Lemma 5.1 (i)]. On the other hand, the formula

$$\mathrm{Map}(f_\sharp(\Lambda_Y)(n)\otimes M,-)=\mathrm{Map}(f_\sharp(\Lambda_Y)(n),\underline{\mathrm{Hom}}(M,-))$$

for $f\colon Y\to X$ smooth and n an integer, combined with the fact that $f_\sharp(\Lambda_Y)(n)$ is compact implies that conditions (3) and (4) are equivalent.

We proved in Corollary 3.6 that motives of geometric origin are constructible. We now prove the converse. Let M be constructible over X. We can assume that X is affine by Zariski hyperdescent of geometric motives. By dévissage and localisation, we can further assume that M is dualisable. In that case $\underline{\mathrm{Hom}}(M,-)$ is compatible with colimits which finishes the proof. □

From this, we deduce continuity for constructible motives.

Corollary 3.8. *Let Λ be a commutative ring which is flat over the integers. Consider a qcqs finite-dimensional scheme X that is the limit of a projective system of qcqs finite-dimensional schemes (X_i) with affine transition maps. Then, the natural map*

$$\mathrm{colim}\ \mathrm{DM}^{\text{ét}}_c(X_i,\Lambda)\to\mathrm{DM}^{\text{ét}}_c(X,\Lambda)$$

is an equivalence.

Proof. We first prove that the functor above is fully faithful. Let (M_i) and (N_i) be cartesian sections over the diagram of schemes (X_i) of the functor $\mathrm{DM}^{\text{ét}}_c(-,\Lambda)$, and let M and N be their respective pullbacks along the maps $X\to X_i$, we want to prove that the canonical map

$$\mathrm{colim}\ \mathrm{Map}(M_i,N_i)\to\mathrm{Map}(M,N)$$

is an equivalence. This can be tested after tensoring with $\mathbb{Q}$ and $\mathbb{Z}/p\mathbb{Z}$ for p any prime number. Thus applying Proposition 3.1, we can assume Λ to be a $\mathbb{Q}$-algebra or a $\mathbb{Z}/p\mathbb{Z}$-algebra. In the case of a $\mathbb{Q}$-algebra, this then follows from Proposition 3.7 and Theorem 3.2 and in the case of a $\mathbb{Z}/p\mathbb{Z}$-algebra, this follows from rigidity [BM21, Theorem 2.2].

We now prove the essential surjectivity of the functor. We use the following result.

Lemma 3.9. *Let Λ be a commutative ring and let M be an object of $\mathrm{DM}^{\text{ét}}_c(X,\Lambda)$ with X a qcqs finite-dimensional scheme. Then, there is a geometric motive P as well as a map $P\to M\oplus M[1]$ whose cone C is such that $C/n\simeq C\oplus C[1]$ for some integer n.*

Proof. Take M a constructible motive. Proposition 3.1 implies that $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\Lambda\otimes\mathbb{Q})$ is the idempotent-completion in $\mathrm{DM}^{\text{ét}}(X,\Lambda\otimes\mathbb{Q})$ of the rationalisation of $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\Lambda)$. The characterisation of the idempotent completion [BG23, Recollection 2.13] as well as Proposition 3.1 yields that $(M\oplus M[1])\otimes\mathbb{Q}$ belongs to the rationalisation of $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\Lambda)$. This gives a geometric motive P and a map $P\to M\oplus M[1]$ which becomes an isomorphism after tensoring with $\mathbb{Q}$. Let C be the cone of this map, then $C\otimes\mathbb{Q}=0$. Using Proposition 3.1 once again, we see that the multiplication by n seen as a map $C\to C$ vanishes for n large enough. This together with the exact triangle $C\xrightarrow{\times n}C\to C/n$ implies that $C/n\simeq C\oplus C[1]$. □

We now finish the proof. Let M be a constructible motive. We want to prove that M is in the essential image of the functor

$$\operatorname{colim} \mathrm{DM}^{\text{ét}}_c(X_i, \Lambda) \to \mathrm{DM}^{\text{ét}}_c(X, \Lambda).$$

Write the exact triangle $P \to M \oplus M[1] \to C$ of the above lemma. Theorem 3.2 implies that P belongs to the essential image. As constructible motives are idempotent-complete, we can assume that M is of the form C/n. But then, this follows from continuity with torsion coefficients which is a combination of the rigidity theorem [BH21, Theorem 3.1] and continuity for constructible étale sheaves [BM21, Theorem 2.2]. □

4 Constructible motives are of geometric origin.

Theorem 4.1. *Let Λ be a commutative ring and let X be a qcqs finite-dimensional scheme. Then*

$$\mathrm{DM}^{\text{ét}}_{\text{gm}}(X, \Lambda) = \mathrm{DM}^{\text{ét}}_c(X, \Lambda).$$

Lemma 4.2. *Let Λ be a ring which is flat over the integers and regular. Let k be a field of characteristic exponent p. Then, the Artin motive functor*

$$\rho_! \colon \mathrm{D}(k_{\text{ét}}, \Lambda[1/p]) \to \mathrm{DM}^{\text{ét}}(k, \Lambda)$$

is a fully faithful monoidal functor. Furthermore any torsion motive lies in its essential image.

Proof. This follows from [Rui22, Corollary 1.2.3.4, Corollary 2.2.2.7]. □

Lemma 4.3. *The theorem holds when X is the spectrum of a field k and Λ be a ring which is flat over the integers and regular.*

Proof. Let M be a constructible motive. We have to show that M is geometric. Lemma 3.9 yields a geometric motive P and a map $P \to M \oplus M[1]$ whose cone C is such that $C \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Furthermore, as C is dualisable, we have

$$\underline{\mathrm{Hom}}(C, \Lambda_k) \otimes_{\mathbb{Z}} \mathbb{Q} = \underline{\mathrm{Hom}}(C, \Lambda_k \otimes_{\mathbb{Z}} \mathbb{Q}) = \underline{\mathrm{Hom}}(C \otimes \mathbb{Q}, \Lambda_k \otimes_{\mathbb{Z}} \mathbb{Q}) = 0.$$

Hence, the dual of C is also a torsion motive.

Writing $C = \rho_! D$, then D is a dualisable object of $\mathrm{D}(k_{\text{ét}}, \Lambda[1/p])$. In particular, it belongs to the thick stable subcategory generated by the $f_\sharp \underline{\Lambda[1/p]}$ for f finite étale according to [Rui22, Proposition 2.1.2.15]. Thus, its image C through $\rho_!$ is geometric as $\rho_!$ is compatible with the functors $f_\sharp$ and f^* for f finite étale using [CD16, §4.4.2]. Finally, $M \oplus M[1]$ is geometric and therefore M is geometric. □

Proposition 4.4. *Theorem 4.1 holds if the ring Λ is flat over the integers and regular.*

Proof. Using Theorem 3.2 and Corollary 3.8, we can assume X to be noetherian. This is now an adaptation of the arguments given in [CD16, Lemma 6.3.25]. Let M be a constructible motive on X. By noetherian induction, using the localisation property and the stability of geometric motives under pushforwards by closed immersions it suffices to show that there is an open immersion $j \colon U \to X$ such that $j^*(M)$ is geometric, *i.e.* we can replace X with any non-empty open subscheme.

Therefore, we can assume X to be irreducible. Let η be the generic point of X. Consider the commutative diagram:

$$\begin{array}{ccc}
\operatorname*{colim}_{U \text{ open}, \, \eta\in U} \mathrm{DM}^{\text{ét}}_{\text{gm}}(U,\Lambda) & \xrightarrow{u} & \operatorname*{colim}_{U \text{ open}, \, \eta\in U} \mathrm{DM}^{\text{ét}}_{c}(U,\Lambda) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle v} \\
\mathrm{DM}^{\text{ét}}_{\text{gm}}(\eta,\Lambda) & \xrightarrow{\beta} & \mathrm{DM}^{\text{ét}}_{c}(\eta,\Lambda).
\end{array}$$

Then, Lemma 4.3 above implies that β is an equivalence and Theorem 3.2 implies that α is an equivalence. On the other hand u is fully faithful and v is an equivalence by Corollary 3.8. Hence, they are both equivalences. The fact that u is an equivalence yields the result. □

The following lemma was considered and proven by Cisinski and Déglise with a particular class of rings.

Lemma 4.5. *Let Λ be a commutative ring and let X be a noetherian scheme and let $M \in \mathrm{DM}^{\text{ét}}_c(X,\Lambda)$, then there exist an étale covering U of X such that the restriction $M_{|U}$ is geometric.*

Proof. The proof is merely an improvement of [CD16, Theorem 6.3.26]. By dévissage and localisation, it is enough to prove the claim for M dualisable. Let x be a point of X, and let $S = \operatorname{Spec} \mathscr{O}^{\text{sh}}_{X,x}$ be the strict henselisation of X at x. The map $u\colon S \to X$ is the limit of the maps $v\colon V \to X$ for V ranging the étale neighborhoods of x in X. For $\mathcal{C}$ a symmetric monoidal ∞-category, denote by $\mathcal{C}^{\text{db}}$ the full subcategory of dualisable objects. The functor

$$\operatorname{colim}_V \mathrm{DM}^{\text{ét}}(V,\Lambda)^{\text{db}} \to \mathrm{DM}^{\text{ét}}(S,\Lambda)$$

is fully faithful. Indeed, given $M, N \in \mathrm{DM}^{\text{ét}}(V_0,\Lambda)$, dualisable, we can compute the following mapping spectra:

$$\mathrm{Map}_{\mathrm{DM}^{\text{ét}}(V,\Lambda)}(v^*M, v^*N) = \mathrm{Map}_{\mathrm{DM}^{\text{ét}}(V,\mathbb{Z})}(\mathbb{Z}_V, v^*(M^\vee \otimes_\Lambda N)),$$

and the same over S if one replaces v by u. The result then follows from [CD16, Remark 6.3.8], but we give the arguments for the reader's convenience. Being an equivalence can be checked after tensoring by $\mathbb{Q}$ and $\mathbb{Z}/p\mathbb{Z}$. The rationalised part is an equivalence because by Proposition 3.1 it is identified with

$$\operatorname{colim}_v \mathrm{Map}_{\mathrm{DM}^{\text{ét}}(V,\mathbb{Q})}(\mathbb{Q}_V, v^*((M^\vee \otimes_\Lambda N)\otimes_{\mathbb{Z}} \mathbb{Q})) \to \mathrm{Map}_{\mathrm{DM}^{\text{ét}}(S,\mathbb{Q})}(\mathbb{Q}_S, u^*((M^\vee \otimes_\Lambda N)\otimes_{\mathbb{Z}} \mathbb{Q})),$$

which is an equivalence because continuity always holds for the presentable ∞-categories and $\mathbb{Q}$ is a compact object of $\mathrm{DM}^{\text{ét}}(V_0,\mathbb{Q})$. The part tensored by $\mathbb{Z}/p\mathbb{Z}$ is an equivalence by rigidity: we are computing étale cohomology in a colimit of topoï, and this holds thanks to [CD16, Lemma 1.1.12 and Theorem 1.1.5].

In particular when applied to the generic points η of X, as $S = \overline{\eta}$ in this case by Proposition 3.7 the image of a dualisable object in $\mathrm{DM}^{\text{ét}}(V,\Lambda)$ in $\mathrm{DM}^{\text{ét}}(\overline{\eta},\Lambda)$ is constructible, hence geometric, thus the functor

$$\operatorname{colim}_V \mathrm{DM}^{\text{ét}}(V,\Lambda)^{\text{db}} \to \mathrm{DM}^{\text{ét}}_{\text{gm}}(\overline{\eta},\Lambda)$$

is an equivalence, and there by the usual properties of spreading out there exists an étale neighborhood V_η of η such that our dualisable object M of X is geometric when restricted to V_η. As X is noetherian, there is an étale covering U of X such that $M_{|U}$ is geometric as an object of $\mathrm{DM}^{\text{ét}}(U,\Lambda)$. □

Proposition 4.6. *Let Λ be a ring of finite type over the integers and let X be a finite-dimensional noetherian scheme. Then Theorem 4.1 holds in* $\mathrm{DM}^{\text{ét}}(X,\Lambda)$.

Proof. By assumption we can write $\Lambda = R/I$ with R a finite type polynomial ring over the integer. By Theorem 4.1 above the result holds over R. Let $M \in \mathrm{DM}^{\text{ét}}_c(X,\Lambda)$. By Lemma 4.5 there exist an étale covering $j : U \to X$ such that j^*M is geometric. In particular, j^*M is a direct factor of a finite colimit of objects of the form $N \otimes_R \Lambda$ with $N \in \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,R) = \mathrm{DM}^{\text{ét}}_c(X,R)$. As R/I is perfect over R, each $N \otimes_R \Lambda$ is in fact constructible as objects of $\mathrm{DM}^{\text{ét}}_c(X,R)$, so that by étale descent $M \otimes_R \Lambda$ is a constructible R-motive, hence geometric, and M is a direct factor of $M \otimes_R \Lambda$. □

From this we deduce the theorem for slightly more general coefficients.

Corollary 4.7. *Let Λ be a ring of finite type over the integers. Then Theorem 4.1 holds for Λ.*

Proof. Assume $X = \lim X_i$ is the limit of X_i finite-dimensional noetherian schemes. By Proposition 4.6 for each X_i we have an equivalence

$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X_i,\Lambda) \simeq \mathrm{DM}^{\text{ét}}_c(X_i,\Lambda).$$

In particular, the canonical functor

$$\mathrm{colim}_i\, \mathrm{DM}^{\text{ét}}_c(X_i,\Lambda) \to \mathrm{DM}^{\text{ét}}_c(X,\Lambda)$$

is fully faithful. The result is therefore equivalent to the essential surjectivity of this functor. For this we may use Lemma 3.9: there exists a geometric object $P \in \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\Lambda)$ and a map $P \to M \oplus M[1]$ with a cofiber C such that there is a $n \in \mathbb{N}^*$ such that $C/n \simeq C \oplus C[1]$. Thus it suffices to know that the constructible object C/n is in the image. As C/n is in the image of the forgetful functor

$$\mathrm{DM}^{\text{ét}}_c(X,\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}) \to \mathrm{DM}^{\text{ét}}_c(X,\Lambda),$$

by Lemma 4.8 it suffices to prove that C/n is in the image of the functor

$$\mathrm{colim}_i\, \mathrm{D}_{\mathrm{cons}}(X_i,\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}) \to \mathrm{D}_{\mathrm{cons}}(X,\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}).$$

Because over noetherian schemes we have proven that constructible and geometric motives coincide with coefficient Λ, modding out by (n) the mapping spectra gives the full faithfulness of the above functor. The claim is that it is essentially surjective. For this, an easy dévissage reduces to the case of a dualisable object K. Let $j : U \to X$ an finite étale covering such that j^*K is constant perfect, this exists thanks to [MW22, Corollary 7.4.12]. Let P be the value of this constant sheaf on U. Then P, seen as a Λ-module, is perfect: it can be written as a finite colimit of free $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ modules, and $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ has a finite resolution by finite free Λ-modules. As $P \otimes_\Lambda (\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})$ has P as a direct factor, we can assume that our P is of the for P'/n for some perfect complex $P' \in \mathrm{Perf}_\Lambda$. In particular, the cosimplicial object $(j^m_\sharp \underline{P'}_{U_m})$ in $\mathrm{D}_{\mathrm{cons}}(X,\Lambda)$ associated to the Čech nerve of j is uniformly bounded, so that its totalisation is finite. This gives that $K = \lim_\Delta j^m_\sharp P'/n$ as a finite limit. Now j, hence all the j^m, are defined over some X_i, so that it is clear that K is pulled back from some X_i. □

The following lemma was used in the proof of the previous corollary.

Lemma 4.8. *Let Λ be a derived commutative $\mathbb{Z}/n\mathbb{Z}$-algebra. Then the natural functor*

$$\mathrm{D}(X_{\text{ét}},\Lambda) \to \mathrm{DM}^{\text{ét}}(X,\Lambda)$$

is an equivalence, which restricts to $\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}},\Lambda) \simeq \mathrm{DM}^{\text{ét}}_c(X,\Lambda)$. Here the definitions one has to take for the objects used in this lemma are exactly the same as the one used everywhere else in this paper.

Proof. The equivalence of presentable ∞-categories is formal: this is true for $\Lambda = \mathbb{Z}/n\mathbb{Z}$, and then we obtain the one we considered by tensoring by Mod_Λ over $\mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}}$. The fact that it restrict to categories of constructible objects follows from the fact that both definition are strictly the same. □

Proof. (of Theorem 4.1) We write $\Lambda = \mathrm{colim}_i \Lambda_i$ with Λ_i a ring of finite type over the integers. For each ring Λ_i we know that $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X, \Lambda_i) = \mathrm{DM}^{\text{ét}}_c(X, \Lambda_i)$. Thus the functor

$$\mathrm{colim}_i\, \mathrm{DM}^{\text{ét}}_c(X, \Lambda_i) \to \mathrm{colim}_i\, \mathrm{DM}^{\text{ét}}_c(X, \Lambda)$$

is fully faithful an the theorem would follow from the fact that it is essentially surjective. By Lemma 3.9 as before we reduce to proving that an object of the form C/n, for $C \in \mathrm{DM}^{\text{ét}}_c(X, \Lambda)$, is in the image, but then this reduces to proving that C/n is in the image of the fully faithful functor

$$\mathrm{colim}_i\, \mathrm{D}_{\text{cons}}(X, \Lambda_i \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}) \to \mathrm{D}_{\text{cons}}(X, \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}).$$

The above functor is indeed fully faithful because for each i we know that

$$\mathrm{D}_{\text{cons}}(X, \Lambda_i \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}) = \mathrm{DM}^{\text{ét}}_c(X, \Lambda_i \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})$$

and $\mathrm{DM}^{\text{ét}}_c(X, \Lambda_i) = \mathrm{DM}^{\text{ét}}_{\text{gm}}(X, \Lambda_i)$, so the full faithfulness follows from continuity of geometric motives, mod n. Now to prove that the functor is essentially surjective, we may assume that the object K we are trying to reach is dualisable, and even of the form $j_\sharp P$ with P perfect constant and $j\colon U \to X$ étale, as in the proof of Corollary 4.7. By [HRS23, Lemma 2.2] there is a P_i perfect over Λ_i such that $P = P_i \otimes_{\Lambda_i} \Lambda$, and the proof is finished. □

Remark 4.9. Let X be a qcqs finite-dimensional scheme and let Λ be a ring of n torsion, with n an integer which is invertible on X. Then, by rigidity, the content of Theorem 4.1 is that the category $\mathrm{D}_{\text{cons}}(X, \Lambda)$ is generated by the $f_\sharp(\Lambda_Y)(n)$ for $f\colon Y \to X$ smooth and n an integer. This result fails in the ℓ-adic setting over a finite field because not all ℓ-adic complexes are mixed in the sense of [BBD82, Section 5].

5 Milnor excision and h-descent.

Proposition 5.1. *Let Λ be a commutative ring. The functor $\mathrm{DM}^{\text{ét}}_{\text{gm}}(-, \Lambda)$ satisfies cdh-descent over qcqs finite-dimensional schemes. Over qcqs schemes of finite valuative dimension, it also satisfies cdh hyperdescent.*

Proof. Since the presentable ∞-category $\mathrm{DM}^{\text{ét}}(-, \Lambda)$ satisfies Nisnevich descent, the localisation property and proper base change, it satisfies cdh-descent as in the proof of [CD19, Theorem 3.3.10] applies in this level of generality. Thus $\mathrm{DM}^{\text{ét}}_{\text{gm}}(-, \Lambda)$ satisfies non effective cdh descent. The effectivity of descent follows from cdh-excision: a descent data of geometric motives gives us a motive as a finite limit of proper pushforwards, which is itself geometric by Corollary 3.5 (1). For hyperdescent, this follows from the fact that the cdh topos is hypercomplete over qcqs schemes of finite valuative dimension by [EHIK21a, Corollary 2.4.16]. □

We now prove that geometric étale motives satisfy h-descent in the sense of Rydh.

Theorem 5.2. *Let Λ be a commutative ring. The functor $\mathrm{DM}^{\text{ét}}_{\text{gm}}(-, \Lambda)$ satisfies h-descent in the sense of [Stacks, Tag 0ETQ] over qcqs finite-dimensional schemes.*

Proof. By [BS17, Theorem 2.9], h-descent is equivalent to cdh-descent combined with fppf-descent. Also, fppf-descent is the same as finite flat descent together with Nisnevich descent, see [GK15, Definition 2.5(15)]. Therefore it suffices to prove that $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\Lambda)$ has descent for the finite flat topology. Let $f\colon T \to S$ be a finite morphism.

By writing $S = \lim_i S_i$ where the S_i are affine $\mathbb{Z}$-schemes of finite type, we can find an index i_0 such that f is the pullback of a finite map $f_{i_0}\colon T_{i_0} \to S_{i_0}$ using [Gro66, Théorème 8.10.5]. By noetherian induction, we can find a finite stratification $\{S^1_{i_0},\dots,S^p_{i_0}\}$ of S_{i_0} such that $S^k_{i_0}$ is open in $\bigcup_{k'\geqslant k} S^{k'}_{i_0}$ for each k and f_{i_0} is the composition of a finite étale map and a purely inseparable map over each $S^k_{i_0}$. Hence, pulling back this stratification to S, we get a stratification $\{S^1,\dots,S^p\}$ of S with the same properties.

First we prove non-effective descent. For this, it suffices to prove that for any objects M and N of $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(S,\Lambda)$, the morphism

$$\mathrm{Map}_{\mathrm{DM}^{\text{ét}}}(M,N) \to \lim_{n\in\Delta} \mathrm{Map}_{\mathrm{DM}^{\text{ét}}}(M,(f_n)_*f_n^*N)$$

is an equivalence, where for any non-negative integer n, we let $f_n\colon T_n \to S$ be the projection given by the Čech nerve of f. Assume first that we have a closed immersion $i\colon Y \to S$ with open complement $j\colon U \to S$ such that the above map is an equivalence for (j^*M, j^*N) and (i^*M, i^*N). Then using that limit of exact triangles are exact triangles we have a morphism of exact triangles of spectra:

$$\begin{array}{ccccc}
\mathrm{Map}(M, j_\sharp j^*N) & \longrightarrow & \mathrm{Map}(M,N) & \longrightarrow & \mathrm{Map}(M, i_*i^*N) \\
\downarrow & & \downarrow & & \downarrow \\
\lim\limits_{n\in\Delta} \mathrm{Map}(M,(f_n)_*f_n^*j_\sharp j^*N) & \longrightarrow & \lim\limits_{n\in\Delta} \mathrm{Map}(M,(f_n)_*f_n^*N) & \longrightarrow & \lim\limits_{n\in\Delta} \mathrm{Map}(M,(f_n)_*f_n^*i_*i^*N)
\end{array}$$

But we have canonical identifications

$$(f_n)_*f_n^*j_\sharp j^*N \simeq j_\sharp(f_n)_*f_n^*j^*N \text{ and } (f_n)_*f_n^*i_*i^*N \simeq i_*(f_n)_*f_n^*i^*N,$$

thus the left and right vertical maps of the above diagram are equivalences by assumption, so that the middle one also. Therefore, by induction on the number p of strata of S, this proves the full faithfulness by induction, given that we have descent along finite étale maps by Lemma 2.4 and that localisation ensures that $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\Lambda)$ is an equivalence along purely inseparable maps.

For the effectivity of descent, we also prove the result by induction. Keeping the same notations as before, we have the following morphism of cofiber sequences of ∞-categories:

$$\begin{array}{ccccc}
\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(Y,\Lambda) & \xrightarrow{i_*} & \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\Lambda) & \xrightarrow{j^*} & \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(U,\Lambda) \\
\downarrow\wr & & \downarrow & & \downarrow\wr \\
\lim\limits_{n\in\Delta} \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(Y_n,\Lambda) & \xrightarrow{\tilde{i_*}} & \lim\limits_{n\in\Delta} \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X_n,\Lambda) & \xrightarrow{\tilde{j^*}} & \lim\limits_{n\in\Delta} \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(U_n,\Lambda)
\end{array}$$

in which the left vertical map is an equivalence by induction, and the right vertical map is an equivalence by étale descent. The functors $\tilde{i_*}$ and $\tilde{j_\sharp}$ have left adjoint $\tilde{i^*}$ and $\tilde{j_\sharp}$ that are computed term wise. In particular as we have a cofiber sequence of stable ∞-categories, in $\lim_\Delta \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X_n,\Lambda)$ we have an exact sequence

$$\tilde{j_\sharp}\tilde{j^*} \to \mathrm{Id} \to \tilde{i_*}\tilde{i^*},$$

which is compatible with the same exact sequence in $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X,\Lambda)$. Thus, if (M_n) is some descent data in $\lim_\Delta \mathrm{DM}^{\text{ét}}_{\text{gm}}(X_n,\Lambda)$, there are objects N_U and N_Y in $\mathrm{DM}^{\text{ét}}_{\text{gm}}(U,\Lambda)$ and $\mathrm{DM}^{\text{ét}}_{\text{gm}}(Y,\Lambda)$ such that they correspond to the restrictions of (M_n) to (U_n) and (Y_n). Thus in $\lim_\Delta \mathrm{DM}^{\text{ét}}_{\text{gm}}(X_n,\Lambda)$ there is a map $\tilde{i_*}N_Y \to \tilde{j_\sharp}N_U[1]$ of fiber (M_n), which by full faithfulness gives a similar map in $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X,\Lambda)$, whose fiber has image (M_n). □

We finish the paper by proving Milnor excision.

Proposition 5.3. *Let Λ be a commutative ring. The functor $\mathrm{DM}^{\text{ét}}_{\text{gm}}(-,\Lambda)$ satisfies Milnor excision ([EHIK21a, Definition 3.2.2]) over qcqs finite-dimensional schemes.*

Proof. First, using [EHIK21a, Lemma 3.2.6] any Milnor square (*i.e.* of the form below, bicartesian, with f affine and i a closed immersion) of qcqs finite-dimensional schemes

$$\begin{array}{ccc} W & \xrightarrow{k} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

can be approximated by Milnor squares of finitely presented $\mathbb{Z}$-schemes (and therefore of noetherian schemes) such that f is of finite type. By continuity (Theorem 3.2), and as filtered colimits are compatible with finite limits in Cat_∞, we can therefore assume our schemes to be noetherian and finite-dimensional and f to be of finite type.

By localization, the family of functors (f^*, i^*) is conservative. Hence, the canonical functor

$$\mathrm{DM}^{\text{ét}}_{\text{gm}}(X,\Lambda) \to \mathrm{DM}^{\text{ét}}_{\text{gm}}(Y,\Lambda) \times_{\mathrm{DM}^{\text{ét}}_{\text{gm}}(W,\Lambda)} \mathrm{DM}^{\text{ét}}_{\text{gm}}(Z,\Lambda)$$

is fully faithful. To show that it is essentially surjective, take geometric motives M_Y, M_Z and M_W over Y, Z and W such that we have equivalences $k^*(M_Y) \simeq M_W \simeq g^*(M_Z)$. Assume first that the ring Λ is flat over the integers. Let M be the fiber of the map $f_*(M_Y) \oplus i_*(M_Z) \to f_*k_*(E_W)$. As our schemes are noetherian the morphisms of type h_* for h of finite type preserve geometric motives using [CD16, Theorem 6.2.13], hence the motive M is constructible. To finish the proof, it suffices to show that the natural maps $f^*(M) \to M_Y$ and $i^*(M) \to M_Z$ are equivalences. Using Proposition 3.1, we can therefore assume Λ to be a $\mathbb{Q}$-algebra or to be a $\mathbb{Z}/p\mathbb{Z}$-algebra for some prime number p. The first case follows from [EHIK21b, Theorem 5.6, Lemma 2.1] while the second case follows from [BM21, Theorem 5.14] and [EHIK21a, Corollary 3.2.11]. Now for a ring Λ of finite type over the integers that we write as R/I with R a polynomial ring, we proceed as follows: as Λ is perfect over R, the three motives M_Y, M_Z and M_W map to geometric objects of $\mathrm{DM}^{\text{ét}}_{\text{gm}}(-,R)$, together with descent data. Thus there exists a $N \in \mathrm{DM}^{\text{ét}}_{\text{gm}}(X,R)$ whose restrictions to Y, Z and W are M_Y, M_Z and M_W. Thus $N \otimes_R \Lambda$ is an object of $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X,\Lambda)$ with an image in the fibre product of ∞-categories containing the data $(M_Y, M_Z, M_W, isos)$ as a direct factor. The full faithfulness implies that $(M_Y, M_Z, M_W, isos)$ is in the image. Now given an arbitrary commutative ring Λ, we may write it as a filtered colimit of finite type rings over the integers, and using continuity the result is therefore a consequence of the commutation of finite limits with filtered colimits. □

Acknowledgments.

The authors would like to thank Denis-Charles Cisinski very warmly for his interest in this work and his help on some technical points that have improved greatly the quality of the paper, in particular for the proofs of Lemma 3.4 and Proposition 4.6. We are grateful to Sophie Morel and

Frédéric Déglise for their support and for useful comments. Finally, RR would like to thank Fabrizio Andreatta, Federico Binda and Alberto Vezzani for some useful conversations.

We were funded by the ANR HQDIAG, RR was funded by the *Prin 2022: The arithmetic of motives and L-functions* and ST had his PhD fundings given by the É.N.S de Lyon.

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