On the Nori and Hodge realisations of Voevodsky étale motives

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Abstract

We show that the derived category of perverse Nori motives and mixed Hodge modules are the derived categories of their constructible hearts. This enables us to construct ∞-categorical lifts of the six operations and therefore to obtain realisation functors from the category of Voevodsky étale motives to the derived categories of perverse Nori motives and mixed Hodge modules that commute with the operations. We also prove that if a motivic t-structure exists then Voevodsky étale motives and the derived category of perverse Nori motives are equivalent. Finally we give a presentation of the indization of the derived category of perverse Nori motives as a category of modules in Voevodsky étale motives.

Contents

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Introduction

Let k be a field of characteristic zero. Following the vision of Beilinson, Deligne, Grothendieck and others there should exist an abelian category of mixed motives $\mathcal{MM}(k)$, target of the universal cohomology theory

$$M^* : \mathrm{Var}_k^{\mathrm{op}} \to \mathcal{MM}(k)$$

on k-varieties in the sense that any other reasonable cohomology theory on k-varieties would factor uniquely through M^*. This is out of reach of the current technology. However two constructions have almost all the required properties to provide a category of mixed motives. The first one is Voevodsky, Levine, Hanamura and others' triangulated category of geometric motives $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ ([VSF00]), a candidate for the bounded derived category of mixed motives. The second is the abelian category of Nori motives $\mathcal{M}_{\mathrm{Nori}}(k)$ ([Fak00]), a candidate for $\mathcal{MM}(k)$. These categories have realisation functors factoring the known cohomology theories of varieties. Indeed, Huber ([Hub00]), Levine ([Lev05]) and Nori have constructed a Hodge realisation functor

$$\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q}) \to \mathrm{D}^b(\mathrm{MHS}_{\mathbb{Q}}^p)$$

to the derived category of polarisable rational mixed Hodge structures that realises the Hodge cohomology of varieties and factors through the derived category of the $\mathbb{Q}$-linear abelian category of motives constructed by Nori thanks to the work of Harrer ([Har16]) and Choudhury–Gallauer ([CGAdS17]). When k is not a necessarily a subfield of $\mathbb{C}$ they also constructed a ℓ-adic realisation with values in the derived category of Galois representations.

There exist relative versions of the categories of Voevodsky and Nori. The triangulated category of constructible rational étale motives $\mathcal{DM}_c(-)$ constructed by Ayoub ([Ayo14a]) and Cisinski and Déglise ([CD16]) gives a triangulated category of étale motivic sheaves generalising Voevodsky's category, and the abelian category of perverse Nori motives $\mathcal{M}_{\mathrm{perv}}(X)$ constructed by Ivorra and S. Morel ([IM22]) gives a category of motivic sheaves generalising the abelian category of Nori. Both settings afford a 6 functors formalism, the tensor product on perverse Nori motives being constructed by Terenzi ([Ter24]). Note that another approach based on constructible sheaves has been considered by Arapura in [Ara13] and [Ara23] but his construction does not include all 6 operations. If X is a quasi-projective smooth k-variety, a realisation functor

$$\mathrm{DM}_c(X) \to \mathrm{D}^b(\mathrm{MHM}(X))$$

to Saito's derived category of mixed Hodge modules ([Sai90]) has been constructed by Ivorra ([Ivo16]). It also factors through the derived category of perverse Nori motives and computes the relative Hodge homology of X-schemes. Unfortunately, Ivorra's functor has no obvious compatibility with the 6 operations that exist on $\mathrm{DM}_c(-)$ and $\mathrm{D}^b(\mathrm{MHM}(-))$. The reason behind this difficulty is that the functor is built by constructing an explicit complex $K^\bullet \in \mathrm{Ch}^b(\mathrm{MHM}(X))$ that computes the relative homology sheaf $f_!f^!\mathbb{Q}_X \in \mathrm{D}^b(\mathrm{MHM}(X))$ of a smooth affine X-scheme $f: Y \to X$, and that we do not know if one can construct the 6 operations explicitly on complexes only in terms of objects living in the abelian category $\mathrm{MHM}(X)$.

On the other hand, we know since Robalo's thesis [Rob15] that the ∞-category $\mathcal{DM}_c(X)$ (which is a stable ∞-category lifting $\mathrm{DM}_c(X)$) affords an universal property that enables one to construct realisations functors to any stable ∞-category $\mathcal{C}$ that have reasonable properties. Indeed, he shows that $\mathcal{DM}(X) = \mathrm{Ind}\mathcal{DM}_c(X)$ is the target of the universal symmetric monoidal functor $\mathrm{Sm}_X \to \mathcal{DM}(X)$ to a stable $\mathbb{Q}$-linear presentably symmetric monoidal ∞-category, satisfying $\mathbb{A}^1$-invariance, étale hyperdescent and $\mathbb{P}^1$-stability. The work of Drew and Gallauer ([DG22]) shows that this universal property works very well in families, so that the ∞-functor

$$\mathcal{DM}: \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$$

is the target of the universal natural transformation $\mathrm{Sm}_{(-)} \Rightarrow \mathcal{DM}$ of functors $\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ sending $f: Y \to X$ in Sch_k, smooth, to a right adjoint and such that $\mathcal{DM}$ is $\mathbb{Q}$-linear, satisfy non-effective étale descent and $\mathbb{A}^1$-invariance, together with $\mathbb{P}^1$-stability. Moreover Cisinski and Déglise (see also Ayoub's [Ayo07, Scholie 1.4.2]) proved in [CD19, Theorem 4.4.25] that the full subcategory of geometric objects $\mathcal{C}_{\mathrm{gm}}$ of any such functor $\mathcal{C}$ affords the six operations, and that any natural transformation $\mathcal{DM} \to \mathcal{C}$ between such functors would induce a natural transformation between the categories of geometric objects that commutes with the six operations. In other words, to construct a family of realisation functors $\mathrm{Hdg}^*: \mathcal{DM}_c \to \mathcal{D}^b(\mathrm{MHM}(-))$ compatible with the 6 operations, it suffices to lift the 6 operations on mixed Hodge modules to the ∞-categorical setting. This is what we do.

There is another approach to the Hodge realisation of étale motives considered by Brad Drew in [Dre18]. For each finite type k-scheme X, he constructs a new category $\mathcal{DH}_c(X)$ of *motivic Hodge modules* that has a tautological functor $\mathcal{DM}_c(X) \to \mathcal{DH}_c(X)$ commuting with all the operations. If $X = \mathrm{Spec}\ k$ is the spectrum of a field, then $\mathcal{DH}_c(\mathrm{Spec}\ k) \hookrightarrow \mathcal{D}^b(\mathrm{MHS}_k^p)$ embeds in the derived category of mixed Hodge structures. Moreover one can compute Deligne cohomology as a Hom-group in $\mathcal{DH}_c(X)$. However it was unclear how to relate $\mathcal{DH}_c(X)$ with $\mathcal{D}^b(\mathrm{MHM}(X))$ as it is hard to construct an adequate t-structure on $\mathcal{DH}_c(X)$. Using our result on the existence of the Hodge realisation and a description of geometric mixed Hodge modules as modules in $\mathcal{DM}(X)$ (see Proposition 4.18 for the case of Nori motives), it should not be too hard to prove that Drew's category $\mathcal{DH}_c(X)$ embeds fully faithfully in the derived category of mixed Hodge modules. We plan to pursue this in a sequel to this paper.

Our main result is the following:

Theorem 0.1 (Theorem 4.2, Theorem 4.4 and Proposition 4.7)**.** *The 6 operations on the categories* $\mathrm{D}^b(\mathrm{MHM}(-))$ *and* $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-))$ *constructed by Saito, Ivorra, S. Morel and Terenzi admit* ∞*-categorical lifts and are defined over any finite type* k*-scheme. For every finite type* k*-scheme there exist* ∞*-functors*

$$\mathrm{Nor}^* : \mathcal{DM}_c(X) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)),$$

$$R_H : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathrm{MHM}(X)),$$

and

$$\mathrm{Hdg}^* : \mathcal{DM}_c(X) \to \mathcal{D}^b(\mathrm{MHM}(X))$$

that commute with the 6 *operations and such that* $R_H \circ \mathrm{Nor}^* \simeq \mathrm{Hdg}^*$*. Moreover,* Nor^* *is compatible with the Betti and the* ℓ*-adic cohomology,* R_H *is* t*-exact and all functors are weight-exact.*

How to give ∞-categorical lifts of triangulated functors? In general this is a hard question. One general solution would be to use the formalism developed by Liu and Zheng in [LZ15]. In our particular case we have a simpler method. Indeed, we use that it is very easy to construct ∞-categorical lifts of derived functors. Most of the 6 operations are not t-exact (even on one side) for the perverse t-structure. However, they are t-exact (at least on one side) for the constructible t-structure, that is the t-structure whose heart behaves like the abelian category of constructible sheaves. In [Nor02], Nori proves that the triangulated category of cohomologically constructible sheaves $\mathrm{D}^b_c(X(\mathbb{C}), \mathbb{Q})$ on the $\mathbb{C}$-points of an algebraic variety X is the derived category of its constructible heart. We show that one can adapt his argument to make things work for mixed Hodge modules and perverse Nori motives, and obtain the following result:

Theorem 0.2 (Corollary 2.19)**.** *Let* X *be a quasi-projective* k*-variety. Denote by* $\mathrm{MHM}_c(X)$ *(*resp. *by* $\mathcal{M}_{\mathrm{ct}}(X)$*) the heart of the constructible* t*-structure on* $\mathrm{D}^b(\mathrm{MHM}(X))$ *(*resp. *on* $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$*). The canonical* ∞*-functors*

$$\mathrm{real}_{\mathrm{MHM}_c(X)} : \mathcal{D}^b(\mathrm{MHM}_c(X)) \to \mathcal{D}^b(\mathrm{MHM}(X))$$

and

$$\mathrm{real}_{\mathcal{M}_{\mathrm{ct}}(X)} : \mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$$

are equivalences of ∞*-categories.*

This theorem enables us to make the handy change of variables $\mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(X)) \simeq \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ and see, for example, pullback functors f^* as derived functors of their restriction to the constructible heart, so that they are ∞-functors for a simple reason! This consideration was the starting idea of this paper. Each of the 6 operations can be obtained as a right derived functor on the constructible or the perverse heart. This is how we obtain the ∞-categorical lifts of the operations.

Let $\mathcal{DN}(X) = \mathrm{Ind}\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ be the indization of the bounded derived ∞-category of $\mathcal{M}_{\mathrm{perv}}(X)$. This is a compactly generated stable presentably symmetric monoidal ∞-category whose compact objects consist of $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$. The functor $\mathrm{Nor}^* : \mathcal{DM}_c(X) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ extends to a functor

$$\mathrm{Nor}^* : \mathcal{DM}(X) \to \mathcal{DN}(X)$$

which preserves colimits, hence has a right adjoint Nor_*. For each finite type k-scheme, denote by $\mathscr{N}_X := \mathrm{Nor}_*\mathrm{Nor}^*\mathbb{Q}_X \in \mathcal{DM}(X)$. As Nor_* is naturally lax symmetric monoidal, we have that $\mathscr{N}_X \in \mathrm{CAlg}(\mathcal{DM}(X))$ is an $\mathbb{E}_\infty$-algebra of $\mathcal{DM}(X)$. Using the proof of the same result for the Betti realisation in [Ayo22, Theorem 1.93], we prove

Proposition 0.3 (Proposition 4.18)**.** *For each scheme X of finite type over some field k of characteristic zero, denote by $p_X : X \to \mathrm{Spec}\ \mathbb{Q}$ the unique morphism. Then $p_X^*\mathscr{N}_{\mathbb{Q}} \simeq \mathscr{N}_X$ and the natural functor*

$$\mathrm{Mod}_{\mathscr{N}_X}(\mathcal{DM}(X)) \to \mathcal{DN}(X)$$

is an equivalence of categories that commutes with pullback functors.

Although we do not go in the details, the same arguments would prove that the ∞-category $\mathcal{DH}^{\mathrm{geo}}(X) \subset \mathrm{Ind}\mathcal{D}^b(\mathrm{MHM}(X))$ of geometric origin objects in the indization of the derived category of mixed Hodge modules is also the category of modules over some algebra $\mathscr{H}_X$ in $\mathcal{DM}(X)$. This shows that Drew's constructions in [Dre18] indeed provide an module presentation of geometric origin mixed Hodge modules (and even, by considering motives enriched in mixed Hodge structures, a presentation of all mixed Hodge modules as modules in a motivic category).

As the functor $(\mathcal{A}, \mathcal{C}) \mapsto \mathrm{Mod}_{\mathcal{A}}(\mathcal{C})$ that sends a symmetric monoidal ∞-category pointed at an algebra object $\mathcal{A} \in \mathrm{CAlg}(\mathcal{C})$ to the ∞-category of modules over $\mathcal{A}$ preserves colimits, we obtain the following corollary:

Corollary 0.4 (Corollary 4.20)**.** *The ∞-functor $X \mapsto \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ has an unique extension $\mathcal{DN}_c(X)$ to quasi-compact and quasi-separated schemes of characteristic zero such that for all limit $X = \lim_i X_i$ of such schemes with affine transitions the natural functor*

$$\mathrm{colim}_i \mathcal{DN}_c(X_i) \to \mathcal{DN}_c(X)$$

is an equivalence of ∞-categories.

Now that a well behaved comparison functor $\mathrm{Nor}^* : \mathcal{DM}_c(X) \to \mathcal{DN}_c(X)$ is available, one wants to see to what extend we can compare both categories. We have a result in that direction, conditional to the existence of a motivic t-structure on $\mathrm{DM}_{\mathrm{gm}}(k)$, an out of reach conjecture that we know imply all standard conjectures in characteristic zero ([Bei12]). Moreover, by the work of Bondarko in [Bon15, Theorem 3.1.4], if such a t-structure exists, then for all finite type k-schemes, we have a perverse and a constructible t-structure on $\mathcal{DM}_c(X)$.

Theorem 0.5 (Theorem 4.11)**.** *Assume that a motivic t-structure exists for all fields of characteristic* 0*. Let X be a finite type k scheme for such a field k. Then the heart of the perverse t-structure of $\mathcal{DM}_c(X)$ is canonically equivalent to $\mathcal{M}_{\mathrm{perv}}(X)$ the category of perverse Nori motives. Moreover, the functor* $\mathrm{Nor}^* : \mathcal{DM}_c(X) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ *is an equivalence of stable ∞-categories. This implies that $\mathcal{DM}_c(X)$ is both the derived category of its perverse and constructible hearts.*

In particular over a field we would have $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q}) \simeq \mathrm{D}^b(\mathcal{MM}(k))$, showing that the "even more optimistic" expectation of [And04, 21.1.8] is in fact no more optimistic than expecting the standard conjectures to be proven.

Organisation of the paper.

After some recollections on motivic constructions in Section 1, in our Section 2 we review Nori's proof in [Nor02] that the category of cohomologically constructible complexes of sheaves on the $\mathbb{C}$-points of a quasi-projective variety X is equivalent to the derived category of constructible sheaves and show that it gives a proof that the derived categories of the perverse and constructible hearts on the categories of mixed Hodge modules (or of perverse Nori motives) are equivalent. The only idea here was to replace local systems with dualisable objects and cohomology of the global sections with $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))}(\mathbb{Q}_X, -[q])$. The main point of the proof goes by showing effaceability of cohomology of the constant object on affine n-spaces, then deduce it for any affine variety thanks to Noether normalisation.

We divided Section 3 in three parts. In the first one we prove general results about the functoriality of the functor $\mathcal{D}^b(-)$ sending an abelian category to its bounded derived ∞-category. In the second part we prove that the 6 operations on mixed Hodge modules and on perverse Nori motives can be lifted to ∞-categorical functors of stable ∞-categories. The third part is devoted to prove étale descent for the derived category of perverse Nori motives.

In Section 4, we use what have been proven on ∞-categorical enhancements to construct the realisation functors. We then prove the result related to the t-structure conjecture. The main argument is the repeated use of the universal properties of $\mathcal{DM}_c(X)$ and $\mathcal{M}_{\mathrm{perv}}(X)$ that force the Nori realisation functor to be an equivalence when $\mathcal{DM}_c(X)$ has a perverse t-structure. Finally we prove that Nori motives are modules over an algebra in $\mathcal{DM}(X)$. The arguments follow more or less Ayoub's proof of the similar statement for the Betti realisation in [Ayo22].

Forthcoming and future work

Using the ∞-categorical enhancement of mixed Hodge modules we will be able to extend them to Artin stacks, together with the operations, t-structures and weights. In joint work with Raphaël Ruimy we will construct an integral version of the derived category of perverse Nori motives. This will give an Abelian category of motivic sheaves with integral coefficients over any finite type k-scheme with k a field of characteristic zero.

Notations and conventions

For the whole paper, k is a field of characteristic $\mathrm{char} k = 0$. When k is a subfield of $\mathbb{C}$, if X is a finite type k-scheme, we denote by $\mathrm{D}^b_c(X, \mathbb{Q})$ the category of constructible complexes of sheaves on $X(\mathbb{C})$. By [Bei87], it is also the derived category of the abelian category of perverse sheaves $Perv(X, \mathbb{Q})$ defined in [BBD82]. For a general field k, we denote by $\mathrm{D}^b_c(X, \mathbb{Q}_\ell)$ the category of $\mathbb{Q}_\ell$-adic étale sheaves on X.

Starting with Section 3, we will use the language of ∞-categories as developed in [Lura] by Lurie, after Joyal's work on quasi-categories. We fix an universe and call small an ∞-category whose mapping spaces and core are in this universe. We denote by Cat_∞ the ∞-category of non necessarily small ∞-categories. If $\mathcal{C}$ is a symmetric monoidal ∞-category, and $A \in \mathcal{C}$ is a commutative algebra, we denote by $\mathrm{Mod}_A(\mathcal{C})$ the category of modules over A in $\mathcal{C}$. For example, the category of $H\mathbb{Q}$-modules in spectra $\mathrm{Mod}_{\mathbb{Q}} := \mathrm{Mod}_{H\mathbb{Q}}(\mathrm{Sp})$ is the unbounded derived category of $\mathbb{Q}$-vector spaces. We will also have to consider the ∞-category of $\mathrm{Mod}_{\mathbb{Q}}$-modules in the ∞-category of stable ∞-categories, that we denote by $\mathrm{Mod}_{\mathbb{Q}}(\mathrm{St})$ and which is equivalent to the category of stable $\mathbb{Q}$-linear ∞-categories. We will also denote by Gpd_∞ the ∞-category of ∞-groupoids, which is also the ∞-category of Kan complexes.

We try to always put the index $\mathcal{C}$ when mentioning Hom, as in $\mathrm{Hom}_{\mathcal{C}}(A, B)$. In the case where $\mathcal{C}$ is an ordinary category, this is the Hom-set, and in the case $\mathcal{C}$ is an ∞-category this is the π_0 of the mapping space $\mathrm{map}_{\mathcal{C}}(A, B)$. In stable categories, the mapping spectra will be denoted by $\mathrm{Map}_{\mathcal{C}}(-, -)$.

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1 Recollections.

1.1 Categories of étale motives.

Ayoub ([Ayo10] and [Ayo14a]), Cisinski-Déglise ([CD19] and [CD16]) and Voevodsky ([Voe02]) have constructed triangulated categories of motivic sheaves, that are the relative versions of $\mathrm{DM}_{\mathrm{gm}}(k)$. In this paper, we are interested in the étale versions. We will present the stable ∞-categorical construction as this is the one we will be using. Also, we only consider $\mathbb{Q}$-linear categories.

We start with Ayoub's construction ([Ayo10]). Let S be a scheme. One starts with the category Sm_S of smooth S-schemes, and consider presheaves of spectra on it : $\mathrm{PSh}(\mathrm{Sm}_S, \mathrm{Sp})$. The category of rational effective étale motivic sheaves is the full subcategory of $\mathrm{PSh}(\mathrm{Sm}_S, \mathrm{Sp})$ whose objects are the presheaves $\mathcal{F}$ that are $\mathbb{Q}$-linear (*i.e.* whose image lands in $\mathcal{D}(\mathbb{Q}) \simeq \mathrm{Mod}_{H\mathbb{Q}} \subset \mathrm{Sp}$), $\mathbb{A}^1$-invariant (*i.e.* the natural map $\mathcal{F}(X) \to \mathcal{F}(\mathbb{A}^1_X)$ is an equivalence for any smooth S-scheme X), and satisfy étale hyper-descent (*i.e.* for any étale hyper cover $U_\bullet \to X$ of a smooth S-scheme X, the natural map $\mathcal{F}(X) \to \lim_\Delta \mathcal{F}(U_n)$ is an equivalence.) It is a symmetric monoidal stable ∞-category, the tensor product being inherited from the Cartesian structure of Sm_S. Inverting the object $\mathrm{cofib}(S \xrightarrow{1} \mathrm{Gm}_S)$ for the tensor product gives the category that we will denote by $\mathcal{DM}(S)$ and call the category of Voevodsky (étale) motives. Note that we could have considered sheaves of spaces instead of spectra if we had inverted $\mathrm{cofib}(S \xrightarrow{\infty} \mathbb{P}^1_S) \simeq \mathrm{cofib}(S \xrightarrow{1} \mathbb{G}_{mS}) \otimes S^1$ and take $\mathbb{Q}$-linear coefficient after this, obtaining an isomorphism $\mathcal{DM}(S) = \mathcal{SH}_{\acute{e}t}(S)_\mathbb{Q}$ with the rational part of the étale stable motivic homotopy category (see [Bac21]). As a formula, we have

$$\mathcal{DM}(S) := (\mathrm{L}_{\mathbb{A}^1}\mathrm{Shv}^\wedge_{\acute{e}t}(\mathrm{Sm}_S, \mathcal{D}(\mathbb{Q})))[(\mathbb{G}_{mS}/1_S)^{-1}].$$

Cisinski and Déglise's construction ([CD16]) follows Voevodsky original construction ([Voe92]). They choose the h-topology instead of the étale topology, whose covers are given by the proper surjective maps. As Sm_S is not big enough to consider this topology, they make the same construction as Ayoub, but starting with finite type S-schemes, to obtain a category $\underline{\mathcal{DM}}_h(S, \mathbb{Q})$:

$$\underline{\mathcal{DM}}_h(S, \mathbb{Q}) := (\mathrm{L}_{\mathbb{A}^1}\mathrm{Shv}^\wedge_h(\mathrm{Sch}_S, \mathcal{D}(\mathbb{Q})))[(\mathbb{G}_{mS}/1_S)^{-1}]$$

The category $\mathcal{DM}_h(S)$ of h-motives is the presentable stable full subcategory generated by the objects of the form $X \otimes \mathbb{Q}(1)^k$ for $X \in \mathrm{Sm}_S$, $\mathbb{Q}(1) = \mathrm{cofib}(S \xrightarrow{1} \mathrm{Gm}_S)[-1]$ and $k \in \mathbb{Z}$. They show in [CD16, Corollary 5.5.7] (one needs [Bac21] to prove the result in the best generality) that the change of sites map induces an equivalence of categories

$$\mathcal{DM}(S) \to \mathcal{DM}_h(S)$$

when S is a Noetherian scheme of finite dimension.

Both categories are stable presentably symmetric monoidal ∞-categories (for a detailed ∞-categorical construction, see [Pre23] or [RS20]). We will denote by $\mathcal{DM}_c(S)$

the full subcategory of étale motives whose objects are compact objects (*i.e.* $M \in \mathcal{DM}(S)$ such that $\mathrm{Hom}_{\mathcal{DM}}(M, -)$ commutes with small filtered colimits.) As we work rationally, one can show ([Ayo14a, Proposition 8.3]) that the subcategory of compact objects coincides with the idempotent complete stable subcategory generated by motives of the form $\mathrm{M}(X)(-n)$ for X a smooth S-scheme and $n \in \mathbb{N}$, where we have denoted by $\mathrm{M}(X)$ the image of $X \in \mathrm{Sm}_S$ by the Yoneda functor. As $\mathcal{DM}(S)$ is compactly generated, we have $\mathrm{Ind}\mathcal{DM}_c(S) = \mathcal{DM}(S)$. Moreover, if $S = \mathrm{Spec}\ k$ is the spectrum of a field, then there is a natural equivalence $\mathrm{ho}(\mathcal{DM}_c(\mathrm{Spec}\ k)) \simeq \mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ between the homotopy category of étale motives and the triangulated now classical category of Voevodsky motives over a field.

The categories $\mathcal{DM}(X)$ can be put together to give a functor $\mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$ from the category of finite type S-schemes (with S a fixed scheme) to the ∞-category of presentable symmetric monoidal ∞-categories. Moreover, this functor $\mathcal{DM}$ is a coefficient system in the sense of Drew and Gallauer (see Definition A.1), and the ∞-category of constructible objects $\mathcal{DM}_c(-)$ is part of a six functors formalism (see [CD19, Section A.5] for a detailed definition), which include that all pullbacks f^* have right adjoints f_*, that there exists exceptional functoriality consisting of a $f_!$ left adjoint to $f^!$, such that $f^! \simeq f^*$ for f étale and $p_* \simeq p_!$ for p proper.

Ayoub constructed in [Ayo10] the Betti realisation of $\mathcal{DM}(S)$ with values in the derived category of sheaves on the analytification S^{an} on S when S is of finite type over a subfield of the complex numbers, which restricts to $\mathcal{DM}_c(S)$ and then lands into $\mathcal{D}_c^b(S(\mathbb{C}), \mathbb{Q})$ the stable category of bounded complexes of sheaves with constructible cohomology. This category is both the derived category of perverse sheaves ([Bei87]) and of the abelian category of constructible sheaves ([Nor02]). One can find in [Ayo14a] and [CD16] the construction of the ℓ-adic realisation functors with values in the derived category of ℓ-adic sheaves. An ∞-categorical version of this functor can be found in [RS20, 2.1.2]. This is a functor $\mathcal{DM}_c(S) \to \mathcal{D}_c^b(S, \mathbb{Q}_\ell)$, the latter category being enhanced using condensed coefficients in [HRS23] where it is denoted by $\mathcal{D}_{\mathrm{cons}}(S, \mathbb{Q}_\ell)$. It also has a version on the big category $\mathcal{DM}(S)$ by taking indization. All the realisations commute with the 6 operations.

1.2 Perverse Nori motives.

Let X be a quasi-projective k-scheme with k a field of characteristic zero. In [IM22], Ivorra and S. Morel introduced an abelian category of perverse Nori motives $\mathcal{M}_{\mathrm{perv}}(X)$. The construction of the category $\mathcal{M}_{\mathrm{perv}}(X)$ goes as follows:

Let X be a quasi-projective k-scheme. Pick your favourite prime number ℓ. We can compose the ℓ-adic realisation

$$\rho_\ell : \mathcal{DM}_c(X) \to \mathcal{D}^b(X_{\acute{e}t}, \mathbb{Q}_\ell)$$

with the perverse truncation functor

$$ {}^{\mathrm{p}}\mathrm{H}^0 : \mathcal{D}^b(X_{\acute{e}t}, \mathbb{Q}_\ell) \to \mathrm{Perv}(X, \mathbb{Q}_\ell)$$

to obtain a homological functor ${}^{\mathrm{p}}\mathrm{H}^0_\ell$ that will factor through $\mathcal{R}(\mathcal{DM}_c(X))$. This latter category is the abelian category of coherent modules over $\mathcal{DM}_c(X)$: it is the full subcategory of additive presheaves $\mathcal{F} : \mathcal{DM}_c(X)^{\mathrm{op}} \to \mathrm{Ab}$ that fit in exact sequences

$$y_M \to y_N \to \mathcal{F} \to 0$$

where $M \to N$ is a morphism in $\mathcal{DM}_c(X)$. Denote by Z_ℓ the kernel of $\widetilde{{}^{\mathrm{p}}\mathrm{H}^0_\ell} : \mathcal{R}(\mathcal{DM}_c(X)) \to Perv(X, \mathbb{Q}_\ell)$, that is the objects mapping to zero. The quotient category $\mathcal{R}(\mathcal{DM}_c(X))/Z_\ell$ is the category of perverse Nori motives $\mathcal{M}_{\mathrm{perv}}(X)$.

By construction, $\mathcal{M}_{\mathrm{perv}}(X)$ has the following universal property: Any homological functor $H : \mathcal{DM}_c(X) \to \mathcal{A}$ to an abelian category $\mathcal{A}$ such that $H(M) = 0$ whenever ${}^{\mathrm{p}}\mathrm{H}^0_\ell(M) = 0$ for all $M \in \mathcal{DM}_c(X)$ will factor uniquely through the universal functor

$$\mathcal{DM}_c(X) \overset{H_{\mathrm{univ}}}{\to} \mathcal{M}_{\mathrm{perv}}(X).$$

As we will see in Lemma 4.9 this construction is the best approximation possible of the heart of the conjectural perverse t-structure on $\mathcal{DM}_c(X)$. A priori this category depends on the chosen ℓ, but one can show, using a continuity argument that this is not the case because for finite type fields (or more generally, for fields embeddable into $\mathbb{C}$), the Betti realisation can also be used to define $\mathcal{M}_{\mathrm{perv}}(X)$.

In fact, every realisation functor existing on $\mathcal{DM}_c(X)$ gives us a realisation functor on $\mathcal{M}_{\mathrm{perv}}(X)$: we have the ℓ-adic realisation R_ℓ and the Betti realisation R_B. If $S = \mathrm{Spec}\ k$, then Ivorra and S. Morel have proven that $\mathcal{M}_{\mathrm{perv}}(\mathrm{Spec}\ k)$ coincide with the category of Nori constructed by Nori, and thus also have an Hodge realisation.

Ivorra and S. Morel show in [IM22, Theorem 5.1] that the derived category of perverse motives $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-))$ is part of a homotopical 2-functor in the sense of Ayoub [Ayo07, Definition 1.4.1]. This means that they constructed four of the six operations, namely for f a morphism of quasi-projective varieties we have $f^*, f_*, f_!, f^!$, together with Verdier duality $\mathbb{D}$, verifying $f^! \simeq \mathbb{D} \circ f^*\mathbb{D}$ and $f_! \simeq \mathbb{D} \circ f_* \circ \mathbb{D}$. In his PhD thesis Terenzi [Ter24] constructed the remaining two operations: tensor product and internal $\mathscr{H}om$. This proves that $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-))$ is part of a full 6 functor formalism. The 6 operations commute with the realisation R_ℓ and R_B that become morphism of symmetric monoidal stable homotopical 2-functors. The construction of the operations of perverse Nori motives is very similar to Saito's construction of mixed Hodge modules.

1.3 Mixed Hodge modules

Let X be a separated finite type $\mathbb{C}$-scheme. In [Sai90], Saito constructed an abelian category $\mathrm{MHM}(X)$ of mixed Hodge modules, together with the six operations on $\mathrm{D}^b(\mathrm{MHM}(X))$. It has a faithful exact functor $\mathrm{MHM}(X) \to Perv(X^{\mathrm{an}}, \mathbb{Q})$ to the abelian category of perverse sheaves on the analytification of X, such that the induced functor on the derived categories commutes with the operations. If $X = \mathrm{Spec}\ \mathbb{C}$, then there is a natural equivalence $\mathrm{MHM}(\mathrm{Spec}\ \mathbb{C}) \simeq \mathrm{MHS}^p_{\mathbb{Q}}$ between polarisable mixed Hodge structures and mixed Hodge modules. One of the most interesting property of mixed Hodge modules is that its *lisse* objects, that is the objects such that the underlying perverse sheaf is a shift of a locally constant sheaf are exactly polarisable variations of mixed Hodge structures. We will not go in details in the construction of Saito's category, and we will mostly need to know that the functor taking a complex of mixed Hodge modules to its underlying complex of perverse sheaves is conservative and commutes with the operations.

2 Adapting Nori's argument to perverse Nori motives.

Hypothesis on the coefficient system.

k is a field of characteristic zero. For this section we will call *variety* an element of a fixed subcategory Var_k of finite type separated k-schemes. For example Var_k could be the full subcategory of quasi-projective k-varieties or of separated reduced k-schemes of finite type. Choose $\mathrm{D} : \mathrm{Var}_k \to \mathrm{Triang}$ your favourite $\mathbb{Q}$-linear triangulated category of coefficients (*e.g.* the derived category of perverse Nori motives [IM22, Theorem 5.1] and [Ter24, Main theorem] or of mixed Hodge modules [Sai90, Theorem 0.1]) with a 6 functors formalism and a conservative Betti realisation $R_B : \mathrm{D} \to \mathrm{D}^b_c$ compatible with the operations. Note that in the mixed Hodge module situation the Betti realisation is nothing more that the functor forgetting the structure of a mixed Hodge module, and is called 'taking the $\mathbb{Q}$-structure" by Saito. It has a constructible t-structure for which R_B is t-exact when $\mathrm{D}^b_c(X(\mathbb{C}), \mathbb{Q})$ is endowed with its natural t-structure of heart the abelian category of constructible sheaves $Cons(X)$. For $X \in \mathrm{Var}_k$ we will call motives the elements of $\mathrm{D}(X)$.

Recall the following definition:

Definition 2.1. The *constructible t-structure* on $\mathrm{D}(X)$ is the unique t-structure on $\mathrm{D}(X)$ that makes all pullback functors t-exact. Explicitly it is the t-structure given by

$$^{\mathrm{ct}}\mathrm{D}^{\leqslant 0}(X) := \{M \in \mathrm{D}(X) \mid \forall x \in X, x^*M \in \mathrm{D}^{\leqslant 0}(\mathrm{Spec}\ \kappa(x))\}$$

and

$$^{\mathrm{ct}}\mathrm{D}^{\geqslant 0}(X) := \{M \in \mathrm{D}(X) \mid \forall x \in X, x^*M \in \mathrm{D}^{\geqslant 0}(\mathrm{Spec}\ \kappa(x))\}.$$

That this indeed gives a t-structure can be proven by gluing, using that a lisse object (see Definition 2.2) is positive (*resp.* negative) for the constructible t-structure if and only it is positive (*resp.* negative) for the perverse t-structure when shifted by the dimension.

Denote by $\mathcal{M}_{\mathrm{ct}}(X)$ the heart of the constructible t-structure on $\mathrm{D}(X)$. As R_B is t-exact and conservative, all known t-exactness results about the 6 functors are true in D. We will call elements of $\mathcal{M}_{\mathrm{ct}}(X)$ *constructible motives* . Denote by $\mathscr{H}\!om(-,-)$ and $\mathbb{Q}_X$ the internal Hom and unit object of $\mathrm{D}(X)$. If k is too big to be embedded in $\mathbb{C}$, then one needs an ℓ-adic realisation instead of a Betti realisation. Except for Lemma 2.6 which is different in both cases, in this section we will say Betti realisation even for big fields k.

We assume that $\mathrm{D}(X)$ has enough structure so that there exists a natural functor $\mathrm{D}^b(\mathcal{M}_{\mathrm{ct}}(X)) \to \mathrm{D}(X)$ (for example if $\mathrm{D}(X)$ is a derived category as shown by Beilinson, Bernstein, Deligne and Gabber in [BBD82, section 3.1] or more generally if $\mathrm{D}(X)$ is the homotopy category of some stable ∞-category $\mathcal{D}(X)$ as proven in [BCKW19, Remark 7.60]), and we assume that the natural functor $\mathrm{D}^b(\mathcal{M}_{\mathrm{ct}}(\mathrm{Spec}\ k)) \to \mathrm{D}(\mathrm{Spec}\ k)$ is an equivalence over the base field. This is true for perverse Nori motives and mixed Hodge modules.

Definition 2.2. Let X be a k-variety. A motive L is a *lisse* object if its Betti realisation is a local system, that is a locally constant bounded complex of finite dimensional $\mathbb{Q}$-vector spaces. We denote $\mathrm{D}^{liss}(X)$ the full subcategory of lisse objects in $\mathrm{D}(X)$.

Remark 2.3. Note that as $\mathrm{D}(X)$ is closed as a tensor category, given an object $M \in \mathrm{D}(X)$, we have a natural candidate for the strong dual of M : it is $\mathscr{H}om(M, \mathbb{Q}_X)$. Therefore, an object M is dualisable if and only if the map

$$\mathscr{H}om(M, \mathbb{Q}_X) \otimes M \to \mathscr{H}om(M, M) \tag{2.3.1}$$

obtained by the $\otimes$-$\mathscr{H}om$ adjunction from the map $\mathrm{ev}_M \otimes \mathrm{Id}_M : \mathscr{H}om(M, \mathbb{Q}_X) \otimes M \otimes M \to M$, is an isomorphism. We will denote by $M^\wedge = \mathscr{H}om(M, \mathbb{Q}_X)$ the dual of M when M is dualisable.

Proposition 2.4. *Let X be a k-variety. An object $L \in \mathrm{D}(X)$ is lisse if and only if it is dualisable and in that case $L \in \mathcal{M}_{\mathrm{ct}}(X)$ if and only if $L^\wedge \in \mathcal{M}_{\mathrm{ct}}(X)$. In particular, for any variety Y and for every object $N \in \mathrm{D}(Y)$, there exists a dense open on which N is dualisable.*

Proof. For the first point, dualisable objects in $\mathrm{D}^b_c(X, \mathbb{Q})$ are exactly local systems by Ayoub's [Ayo22, Lemma 1.24] , and the Betti realisation is conservative, so we can test if the map (2.3.1) is an isomorphism after applying Betti realisation. If $L \in \mathcal{M}_{\mathrm{ct}}(X)$ is dualisable, the functor $\mathscr{H}om(L, -) = - \otimes L^\wedge$ is t-exact because it is left and right adjoint to the exact functor $- \otimes L$. Therefore, $L^\wedge = \mathscr{H}om(L, \mathbb{Q}) \in \mathcal{M}_{\mathrm{ct}}(X)$ lies in the heart.

The second point follows from the fact that for $N \in \mathrm{D}(Y)$, the complex of sheaves $R_B(N)$ is constructible so there exists a dense open U of Y on which $R_B(N)$ is locally constant. □

2.1 Cohomology of motives over an affine variety.

Definition 2.5. 1. An additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is called *effaceable* if for any $M \in \mathcal{A}$ there is an injection $M \to N$ in $\mathcal{A}(X)$ such that the induced map $F(M) \to F(N)$ is the zero map.

2. An object $M \in \mathrm{D}(X)$ is called *admissible* if the functor $\mathrm{Hom}_{\mathrm{D}(X)}(M, -[q]) : \mathcal{M}_{\mathrm{ct}}(X) \to \mathrm{Vect}_{\mathbb{Q}}$ is effaceable for every $q > 0$.

Lemma 2.6. *Let X be a variety and M be a constructible motive on $\mathbb{A}^1_X$. Assume that there is $U \subset \mathbb{A}^1_X$ a smooth open subset on which M is lisse and such that the restriction of $\pi : \mathbb{A}^1_X \to X$ to $Z = \mathbb{A}^1_X \setminus U$ is finite. Assume that $M_{|Z} = 0$. Then there exists $N \in \mathcal{M}_{\mathrm{ct}}(\mathbb{A}^1_X)$ such that $\pi_* N = 0$ and M injects into N.*

Proof. Assume that k is a subfield of $\mathbb{C}$. Then the following holds : Denote by $p_i : \mathbb{A}^2_X \to \mathbb{A}^1_X$ the projections, and by $\Delta : \mathbb{A}^1_X \to \mathbb{A}^2_X$ the diagonal. The map $\alpha : p_1^* M \to \Delta_* M$ in $\mathcal{M}_{\mathrm{ct}}(X)$ (both functors are t-exact) obtained by adjunction from $\mathrm{id} : \Delta^* p_1^* M \simeq M \to M$ is surjective because it is surjective after realisation. We therefore have an exact sequence

$$0 \to \ker \alpha \to p_1^* M \to \Delta_* M \to 0.$$

Denote $N := \mathrm{H}^1 (p_2)_* \ker \alpha$. Then the long exact sequence of cohomology obtained after applying $(p_2)_*$ to the above sequence gives a map $M = (p_2)_* \Delta_* M = \mathrm{H}^0((p_2)_* \Delta_* M) \to N$, which is injective by [Nor02, Proposition 2.2].

Moreover, Nori proves in *loc cit.* that for all $p \geqslant 0$ we have $\mathrm{H}^p(\pi_* N) = 0$, which is equivalent to $\pi_* N = 0$.

In the case k is too big to be embedded in $\mathbb{C}$, on may use [Bar21, Proposition 3.2] and the ℓ-adic realisation, noting that the construction of N is the same. □

Nori's method gives in our setting a slightly less good result than in the case of constructible sheaves. It will thankfully be sufficient for us.

Theorem 2.7 (*cf.* [Nor02] Theorem 1)**.** *For every $n \in \mathbb{N}$, the motive $\mathbb{Q}_{\mathbb{A}^n_k} \in \mathcal{M}_{\mathrm{ct}}(\mathbb{A}^n_k)$ is admissible.*

Proof. We argue as in Nori's paper.
We proceed by induction on n to prove the theorem. First the case of a point. If $n = 0$ and $M \in \mathcal{M}_{\mathrm{ct}}(\mathrm{Spec}\ k)$, as $\mathrm{D}(\mathrm{Spec}\ k) = \mathrm{D}^b(\mathcal{M}_{\mathrm{ct}}(\mathrm{Spec}\ k))$, the functors $\mathrm{Hom}_{\mathrm{D}(\mathrm{Spec}\ k)}(M, -[q])$ are effaceable by [BBD82, Proposition 3.1.16].

Suppose that the result is known for $n-1$. Let $M \in \mathcal{M}_{\mathrm{ct}}(\mathbb{A}^n_k)$ and $q > 0$. Let $\pi : \mathbb{A}^n_k \to \mathbb{A}^{n-1}_k$ be the projection on the $n-1$ first coordinates. Up to change of coordinates, one can choose a smooth open of the form $D(f)$ in $\mathbb{A}^n_k$ such that M is lisse on that open and the natural projection $\pi_{|V(f)} : V(f) \to \mathbb{A}^{n-1}_k$ is finite. Denote by j and i the inclusions of $D(f) = U$ and $V(f) = Z$ in $\mathbb{A}^n_k$.
By Lemma 2.6, $j_! M_{|U}$ is a submotive of a constructible motive N' such that $\pi_* N' = 0$. Taking N to be the push-out of the maps $j_! M_{|U} \to M$ and $j_! M_{|U} \to N'$ we get a morphism of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_! M_{|U} & \longrightarrow & M & \longrightarrow & i_* M_{|Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & i_* M_{|Z} & \longrightarrow & 0 \end{array}$$

with γ also injective.

Also, as $\pi_*N' = 0$, we see that $\pi_*N \to \pi_*i_*M_{|Z} = (\pi_{|Z})_*M_{|Z}$ is an isomorphism because of the distinguished triangle $N' \to N \to i_*M_Z \xrightarrow{+1}$. As $\pi_{|Z}$ is finite, its pushforward is t-exact, hence $\pi_*N \in \mathcal{M}_{\mathrm{ct}}(\mathbb{A}^{n-1}_k)$.

By the induction hypothesis, we can find an injection $g : \pi_*N \to K$ with K a constructible motive such that $\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, g[q]) = 0$. Take L the pushout of the map $\pi^*\pi_*N \to N$ and the injection (π^* is t-exact) $\pi^*g : \pi^*\pi_*N \to \pi^*K$. We get a morphism of exact sequences :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\pi_*N & \xrightarrow{\pi^*g} & \pi^*K & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \mathrm{id}_C} & & \\ 0 & \longrightarrow & N & \xrightarrow{h} & L & \longrightarrow & C & \longrightarrow & 0 \end{array} . \tag{2.7.1}$$

The unit map $\pi_*N \to \pi_*\pi^*\pi_*N$ is an isomorphism by $\mathbb{A}^1$-invariance. Therefore in the diagram above the left and the right vertical maps become isomorphisms after applying π_*, thus this is also the case for the middle map $\iota : \pi^*K \to L$.

Using the adjunction (π^*, π^*) and the fact that $\pi_*N \to \pi_*\pi^*\pi_*N$ is an isomorphism one gets that the map

$$\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, \pi^*\pi_*N[q]) \to \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, N[q]) \tag{2.7.2}$$

is an isomorphism. Moreover the map

$$\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, \pi^*\pi_*N[q]) \to \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, \pi^*K[q]) \tag{2.7.3}$$

induced by π^*g vanishes as it was already the case before applying π^*.

Therefore when applying $\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, -[q])$ to the left square of (2.7.1), we see that the map

$$\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, N[q]) \to \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, L[q]) \tag{2.7.4}$$

induced by h factors through the zero map.

Hence we see that the injection $M \xrightarrow{\gamma} N \xrightarrow{h} L$ induces the zero map after applying the functor $\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, -[q])$, finishing the induction. $\square$

Corollary 2.8. *If X is affine, the object $\mathbb{Q}_X$ is admissible.*

Proof. Let $q > 0$. Take $i : X \to \mathbb{A}^n_k$ a closed immersion and $M \in \mathcal{M}_{\mathrm{ct}}(X)$. By Theorem 2.7, there is an injection $i_*M \to K$ with K a constructible motive such that $\mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n)}(\mathbb{Q}_{|\mathbb{A}^n_k}, i_*M[q]) \to \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n)}(\mathbb{Q}_{|\mathbb{A}^n_k}, K[q])$ is the zero map. We have an inclusion $f : M \to N := i^*K$, and the image of the map in the Hom-sets is

$$\mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{Q}_X, i^*i_*M[q]) \simeq \mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{Q}_X, M[q]) \to \mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{Q}_X, i^*K[q])$$

which fits in a commutative diagram :

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{Q}_X, M[q]) & \xrightarrow{f\circ -} & \mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{Q}_X, i^*K[q]) \\ {\scriptstyle adj}\uparrow & & {\scriptstyle adj}\uparrow \\ \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, i_*M[q]) & \xrightarrow{0} & \mathrm{Hom}_{\mathrm{D}(\mathbb{A}^n_k)}(\mathbb{Q}_{\mathbb{A}^n_k}, K[q]) \end{array}$$

giving the result. □

2.2 Admissibility of constant motives on any variety.

Notation 2.9. *If $j : U \hookrightarrow X$ is an open subset of a variety X, and $M \in \mathcal{M}_{\mathrm{ct}}(X)$ is a constructible motive, we will use the notation $M[U] := j_! M_{|U}$. Note that by localisation we have $M/M[U] = i_* M_{|Z}$ with $i : Z = X \setminus U \to X$ the closed complement.*

Lemma 2.10 (Stability of admissibility)**.** *Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of constructible motives on X.

1. *Assume that M'' is admissible and at least one of M', M is admissible. Then all three are admissible.*

2. *Assume that M' and M are admissible. If the functor*

$$\mathrm{coker}\ (\mathrm{Hom}_{\mathrm{D}(X)}(M, -) \to \mathrm{Hom}_{\mathrm{D}(X)}(M', -))$$

   is effaceable. Then M'' is admissible.

Proof. These are formal properties of abelian categories, proven in [Nor02, Lemma 3.2]. □

Corollary 2.11 ([Nor02] Lemma 3.5)**.** *Let $M \in \mathcal{M}_{\mathrm{ct}}(X)$*

1. *If $U \subset X$ is open, and if M and $M[U]$ are admissible, then so is $M/M[U]$.*

2. *If V, W are open of X, and if $M[V], M[W], M[V \cap W]$ are admissible, then the same is true for $M[V \cup W]$*

Proof. We use part 2. of Lemma 2.10. For 1., it suffices to show that any $P \in \mathcal{M}_{\mathrm{ct}}(X)$ is a subsheaf of a $Q \in \mathcal{M}_{\mathrm{ct}}(X)$ such that the map $\mathrm{Hom}_{\mathrm{D}(X)}(M, Q) \to \mathrm{Hom}_{\mathrm{D}(X)}(M[U], Q)$ is surjective. Denote by $j : U \to X$ the open immersion and by $i : Z \to X$ its closed complement. Then define $Q = i_*i^*P \oplus j_*j^*P$. Localisation ensures that the map $P \to Q$ is injective, and the induced map between Hom's is

$$\begin{array}{c} \mathrm{Hom}_{\mathrm{D}(X)}(M, i_*P_{|Z}) \oplus \mathrm{Hom}_{\mathrm{D}(U)}(M_{|U}, P_{|U}) \simeq \mathrm{Hom}_{\mathrm{D}(X)}(M, Q) \\ \downarrow \\ \mathrm{Hom}_{\mathrm{D}(U)}(M_{|U}, P_{|U}) \simeq 0 \oplus \mathrm{Hom}_{\mathrm{D}(X)}(j_!M_{|U}, j_*P_{|U}) \simeq \mathrm{Hom}_{\mathrm{D}(X)}(M[U], Q) \end{array}$$

is surjective as it is just the second projection.
For 2. first one has the Mayer-Vietoris exact sequence

$$0 \to M[V \cap W] \to M[V] \oplus M[W] \to M[V \cup W] \to 0. \tag{2.11.1}$$

For a given P we take the same Q as above for $U = V \cap W$ that gives surjectivity of $\mathrm{Hom}_{\mathrm{D}(X)}(M, Q) \to \mathrm{Hom}_{\mathrm{D}(X)}(M[U], Q)$. Now the map $a : M[U] \to M$ factors through $M[V]$ thus the map

$$\mathrm{Hom}_{\mathrm{D}(X)}(M[V] \oplus M[W], Q) \overset{a^* \oplus 0}{\to} \mathrm{Hom}_{\mathrm{D}(X)}(M[U], Q) \tag{2.11.2}$$

is surjective. Finally the map induced by $M[U] \to M[V] \oplus M[W]$ is also surjective as one of its direct summands is. □

Proposition 2.12 (Avoiding the use of injective objects, after Nori [Nor02] Remark 3.8)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories, and let $G : \mathcal{B} \to \mathcal{A}$ be a right adjoint to F. Let $H : \mathcal{A} \to \mathcal{C}$ be an effaceable functor. Then HG is also effaceable.*

Proof. Take B an object of $\mathcal{B}$. By assumption on H, there is an injection $u : G(B) \to A$ such that the induced map $H(u) : HG(B) \to H(A)$ vanishes. By exactness of F, the morphism $F(u) : FG(B) \to F(A)$ is a monomorphism. We also have the counit of the adjunction $\varepsilon : FG(B) \to B$. Take B' the pushout of these two maps :

$$\begin{array}{ccc} FG(B) & \xrightarrow{F(u)} & F(A) \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle t} \\ B & \xhookrightarrow[v]{} & B' \end{array}$$

The map $v : B \to B'$ is a monomorphism and we have a commutative diagram (with η the unit of the adjunction) :

$$\begin{array}{ccc} G(B) & \xhookrightarrow{u} & A \\ {\scriptstyle \eta_{G(B)}}\downarrow & & \downarrow{\scriptstyle \eta_A} \\ GFG(B) & \xrightarrow{GF(u)} & GF(A) \\ {\scriptstyle G(\varepsilon)}\downarrow & & \downarrow{\scriptstyle G(t)} \\ G(B) & \xrightarrow{G(v)} & G(B') \end{array} \qquad (\mathrm{Id}: G(B) \to G(B))$$

The composition $G(\varepsilon) \circ \eta$ is the identity, therefore, the map $G(v)$ factors as $w \circ u$ with $w = G(t) \circ \eta$. Now, $HG(v) = H(w) \circ H(u) = 0$ hence HG is effaceable. □

Corollary 2.13. *Let $F : \mathrm{D}(X) \to \mathrm{D}(Y)$ be a t-exact exact functor which is a left adjoint. If $M \in \mathcal{M}_{\mathrm{ct}}(X)$ is admissible, then so is $F(M) \in \mathcal{M}_{\mathrm{ct}}(Y)$.*

Proof. Let $q > 0$ and let G be a right adjoint of F. The functor $H = \mathrm{Hom}_{\mathrm{D}(X)}(M, -[q])$ is effaceable by definition. By Proposition 2.12, $H \circ G$ is therefore effaceable. We have a natural isomorphism $H \circ G \simeq \mathrm{Hom}_{\mathrm{D}(Y)}(F(M), -[q])$ which gives the claim. $\square$

Corollary 2.14. *Let $M \in \mathcal{M}_{\mathrm{ct}}(X)$ be an admissible motive.*

1. *Let $j : X \to Y$ be an étale morphism. Then $j_!M$ is admissible.*

2. *Let $i : X \to Y$ be a finite morphism. Then i_*M is admissible.*

3. *Let $L \in \mathcal{M}_{\mathrm{ct}}(X)$ be a lisse motive. Then $M \otimes L$ is admissible.*

Proof. We use Corollary 2.13. For the first two points, we have the adjunctions $(j_!, j^*)$ and $(i_*, i^!)$ and $j_!$ and i_* are t-exact. Let $L \in \mathcal{M}_{\mathrm{ct}}(X)$ be a lisse motive. The functor $-\otimes L$ is left adjoint to $\mathscr{H}om(L, -)$ and right adjoint to $\mathscr{H}om(L^\wedge, -)$ hence is exact. $\square$

Proposition 2.15 ([Nor02] Proposition 3.6)**.** *Let $U \subset X$ an open of a variety X. Then the constructible motive $\mathbb{Q}_X[U]$ is admissible.*

Proof. Let $j : U \to X$ be an open immersion. We prove the theorem by induction on the number n of affines needed to cover U. If U is affine, Corollary 2.8 ensures that $\mathbb{Q}_U$ is admissible hence by Corollary 2.14, $j_!\mathbb{Q}_U = \mathbb{Q}_X[U]$ is admissible. For the induction, one can write $U = V \cup W$ with V affine and W covered by $n-1$ affines, and separateness of X gives that $V \cap W$ is covered by $n-1$ affines. Therefore by induction $\mathbb{Q}_X[V]$, $\mathbb{Q}_X[V \cap W]$ and $\mathbb{Q}_X[W]$ are admissible. Then, Corollary 2.11 ensures that $\mathbb{Q}_X[U]$ is admissible. $\square$

Corollary 2.16 ([Nor02] Theorem 2)**.** *Let X be a variety over k. Then the constant motive $\mathbb{Q}_X$ is admissible, that is, for every $q > 0$, every constructible motive $M \in \mathcal{M}_{\mathrm{ct}}(X)$ can be embedded in a constructible motive $N \in \mathcal{M}_{\mathrm{ct}}(X)$ such that the map*

$$\mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{Q}_X, M[q]) \to \mathrm{Hom}_{\mathrm{D}(X)}(\mathbb{Q}_X, N[q])$$

vanishes.

Proof. One takes $U = X$ in the previous proposition. $\square$

2.3 Lisse motives enter the party.

Proposition 2.17. *Let X be a variety. Any lisse motive in $\mathcal{M}_{\mathrm{ct}}(X)$ is admissible.*

Proof. By Corollary 2.16 the unit object $\mathbb{Q}_X$ on X is admissible. By Corollary 2.14 $L \simeq \mathbb{Q}_X \otimes L$ is admissible. $\square$

Theorem 2.18. *Let X be a variety. Then any $M \in \mathcal{M}_{\mathrm{ct}}(X)$ is admissible, that is, for every $q > 0$, the functor $\mathrm{Hom}_{\mathrm{D}(X)}(M, -[q]) : \mathcal{M}_{\mathrm{ct}}(X) \to \mathrm{Vect}_{\mathbb{Q}}$ is effaceable.*

Proof. We prove the result on Noetherian induction on the support S of M in X, that is the complement of the largest open V of X such that $M_{|V} = 0$. Denote by $i : S \to X$ the closed immersion so that $i_* i^* M \simeq M$ by the localisation triangle. If the support of M is empty, then $M = 0$ is admissible. Otherwise note that by Corollary 2.14, if $i^* M$ is admissible then $i_* i^* M$ is also admissible so that we can assume that $S = X$ and $M = i^* M$. By Proposition 2.4 there exists a dense open U of X such that if one denotes by $j : U \to X$ the open immersion the motive $j^* M$ is lisse. Therefore by Proposition 2.17 and Corollary 2.14 the subobject object $j_! j^* M$ of M is admissible. The support of the quotient $M/j_! j^* M$ is strictly smaller than the support of M (because U is nonempty), hence by induction hypothesis it is admissible. The first point of Lemma 2.10 enables us to conclude that M is admissible. □

Corollary 2.19. *The natural functor* $\mathrm{D}^b(\mathcal{M}_{\mathrm{ct}}(X)) \to \mathrm{D}(X)$ *is an equivalence.*

Proof. We apply [BBD82, Lemme 3.1.16]. The conditions are verified because of the previous Theorem 2.18. □

3 Higher categorical enhancements.

3.1 Categorical preliminaries.

Recall the following fundamental definition of Lurie:

Definition 3.1. An ∞-category $\mathcal{C}$ is *stable* if it has all finite limits and colimits, if the initial objects coincides with the final object and if any commutative square in $\mathcal{C}$ is cocartesian if and only if it is Cartesian. An exact functor between stable ∞-categories is an ∞-functor preserving finite limits and colimits.

The homotopy category $\mathrm{ho}(\mathcal{C})$ is always a triangulated category and exact functors induce triangulated functors.

Definition 3.2. The bounded derived ∞-category of an abelian category $\mathcal{A}$ is the ∞ category $\mathcal{D}^b(\mathcal{A})$ obtained by inverting quasi-isomorphisms in the category $\mathrm{Ch}^b(\mathcal{A})$. It is a stable ∞-category with a t-structure, and its homotopy category is the usual derived category $\mathrm{D}^b(\mathcal{A})$.

We will use the following result of Bunke, Cisinski, Kasprowski, and Winges:

Theorem 3.3. *Let* $\mathcal{C}$ *be a stable* ∞*-category and let* $\mathcal{A}_1, \dots, \mathcal{A}_n$ *be abelian categories. Denote by* $\mathrm{Fun}^{\mathrm{nex}}(\prod_i \mathcal{D}^b(\mathcal{A}_i) \to \mathcal{C})$ *the category of* n*-multi-exact functors* $\prod_i \mathcal{D}^b(\mathcal{A}_i) \to \mathcal{C}$ *that are exact in each variables (meaning that if one fixes one variable then we obtain an* $(n-1)$*-multi-exact functor). Denote also by* $\mathrm{Fun}^{\amalg,\mathrm{nex}}(\prod_i \mathcal{A}_i \to \mathcal{C})$ *the category of finite coproduct preserving* n*-multi-exact functors, that is functors* $\prod_i \mathcal{A}_i \to \mathcal{C}$ *that* *are finite coproduct preserving and that in each variable i send short exact sequences $0 \to x \to y \to z \to 0$ to cocartesian squares*

$$\begin{array}{ccc} f(x) & \longrightarrow & f(y) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & f(z) \end{array}$$

(so fixing one variable gives a coproduct preserving $(n-1)$-multi-exact functor).

Then the restriction to the hearts functor

$$\mathrm{Fun}^{\mathrm{nex}}(\prod_i \mathcal{D}^b(\mathcal{A}_i) \to \mathcal{C}) \to \mathrm{Fun}^{\mathrm{II,nex}}(\prod_i \mathcal{A}_i \to \mathcal{C}) \tag{3.3.1}$$

is an equivalence of ∞-categories.

Proof. We give the proof for $n = 2$, the general case being strictly the same except for a too large number of indices. Recall that we have canonical equivalence of ∞-categories $\mathrm{Fun}(\mathcal{D}^b(\mathcal{A}_1) \times \mathcal{D}^b(\mathcal{A}_2), \mathcal{C}) \simeq \mathrm{Fun}(\mathcal{D}^b(\mathcal{A}_1), \mathrm{Fun}(\mathcal{D}^b(\mathcal{A}_2), \mathcal{C}))$. By definition, the ∞-category $\mathrm{Fun}^{2\mathrm{ex}}(\mathcal{D}^b(\mathcal{A}_1) \times \mathcal{D}^b(\mathcal{A}_2) \to \mathcal{C})$ consists exactly of functors that are sent in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}(\mathcal{A}_1), \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^b(\mathcal{A}_2), \mathcal{C}))$ by that equivalence. By [BCKW19, Corollary 7.56] the latter category is exactly $\mathrm{Fun}^{\mathrm{II,ex}}(\mathcal{A}_1, \mathrm{Fun}^{\mathrm{II,ex}}(\mathcal{A}_2, \mathcal{C}))$, which again is equivalent to $\mathrm{Fun}^{\mathrm{II,2ex}}(\mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{C})$. This finishes the proof. □

We will mostly use the particular case of $n = 1$:

Corollary 3.4 ([BCKW19])**.** *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{A}$ be an abelian categories. Then restriction to the heart gives an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^b(\mathcal{A}), \mathcal{C}) \to \mathrm{Fun}^{\mathrm{II,ex}}(\mathcal{A}, \mathcal{C}).$$

Recall that by [Lura, Tag 01F1], we have the following proposition :

Proposition 3.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Assume that for a collection $A \subset \mathcal{C}$ of objects of $\mathcal{C}$ is given the data, for each $a \in A$ of an isomorphism $f_a : F(a) \to d_a$ with $d_a \in \mathcal{D}$. Then there exist a functor $G : \mathcal{C} \to \mathcal{D}$ and a natural equivalence $g : F \Rightarrow G$ such that for every $a \in A$, we have $g_a = f_a$ (and this is really an equality).*

Proof. Indeed, denote by $\iota : A \to \mathcal{C}$ the canonical monomorphism of simplicial sets. Then by [Lura, Tag 01F1], the restriction

$$\iota^* : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(A, \mathcal{D}) \tag{3.5.1}$$

is an isofibration of simplicial sets (*cf.* [Cis19, 3.3.15].) The data of the f_a defines an isomorphism γ in $\mathrm{Fun}(A, \mathcal{D})$ between $F_{|A}$ and the map of simplicial sets $A \to \mathcal{D}$ defined by $a \mapsto d_a$. By the lifting property of isofibrations, there is a functor $G : \mathcal{C} \to \mathcal{D}$ and an isomorphism $F \simeq G$, that is a natural equivalence $g : F \Rightarrow G$ such that $\iota^* g = \gamma$, which is exactly the statement of the proposition. □

We will also need a result on the monoidal structure of the bounded derived category.

Proposition 3.6. *Let $\mathcal{A}$ be an abelian category endowed with a tensor product which is exact in each variable. Then there is a canonical symmetric monoidal structure on the stable ∞-category $\mathcal{D}^b(\mathcal{A})$ such that the inclusion functor $\mathcal{A} \to \mathcal{D}^b(\mathcal{A})$ is symmetric monoidal. Moreover, for any category I with finite coproducts and any functor $\mathcal{A} : I \to$ $\mathrm{SymMono}_1$ of such abelian categories with exact symmetric monoidal transitions, there is a canonical lift of the functor $\mathcal{D}^b \circ \mathcal{A}\colon I \to \mathrm{Cat}_\infty$ to a functor $I \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$.*

Proof. The proof of this proposition is quite easy, but a bit technical. We use that by [BCKW19, Proposition 7.55] for every abelian category $\mathcal{B}$ the ∞-category of complexes $\mathcal{K}^b(\mathcal{B})$ is isomorphic to the Spanier-Whitehead stabilisation $\mathrm{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{B}))$ of smallest stable ∞-category $\mathcal{P}_{\Sigma,f}(\mathcal{B})$ of the ∞-category of additive presheaves of spaces $\mathcal{P}_\Sigma(\mathcal{B})$ that contains the image of the Yoneda embedding. Then inverting quasi-isomorphisms gives $\mathcal{D}^b(\mathcal{B})$. The idea to make this work in families is that the construction $\mathcal{P}_\Sigma(-)$ is a special case of a very general result of Lurie, that enables us to treat the case of a family $\mathcal{A} : I \to \mathrm{CatAb}$ of abelian categories exactly as in the case of one object, by replacing $\mathcal{B}$ with the $\mathscr{O}$-monoid $\mathfrak{A}$ associated to the symmetric monoidal functor $\mathcal{A}$ (see below).

Let $\mathcal{A} : I \to \mathrm{SymMono}_1$ be a diagram of symmetric monoidal abelian categories and symmetric monoidal exact functors. By Barkan's [Bar22, Corollary A] or Lurie's [Lur22, Corollary 5.1.1.7], the forgetful functor induces an equivalence of 2-categories

$$\mathrm{CAlg}(\mathrm{Cat}_1) \to \mathrm{SymMono}_1$$

thus we can see $\mathcal{A}$ as a functor

$$I \to \mathrm{CAlg}(\mathrm{Cat}_1) \subset \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

Let us endow I with the cocartesian symmetric monoidal structure and let $\mathcal{O} \to \mathrm{Fin}_*$ be the associated cocartesian fibration, making it a ∞-operad. By [DG22, Proposition A.11 and Corollary A.12] the above data of $\mathcal{A}$ is equivalent to a $\mathcal{O}$-monoid $\mathfrak{A} \to \mathcal{O}$. Our goal is to derive this $\mathcal{O}$-monoid fiber-wise to obtain a $\mathcal{O}$-monoidal functor $\mathfrak{A} \to \mathfrak{D}$ that will give the symmetric monoidal structure on the functor $\mathcal{D}^b \circ \mathcal{A} : I \to \mathrm{Cat}_\infty$ and the natural transformation $\mathcal{A} \to \mathcal{D}^b \circ \mathcal{A}$.

By [Lur22, Proposition 4.8.1.10] and the fact that $\mathcal{P}_\Sigma(\mathcal{B})$ is the universal recipient of a functor from $\mathcal{B}$ to a ∞-category that have all sifted colimits ([Lur09, Proposition 5.5.8.15]) we deduce the existence of a $\mathcal{O}$-monoidal functor

$$\mathfrak{A} \to \mathfrak{P}(\mathfrak{A})$$

such that the restriction to I of the unstraightening of the fibration $\mathfrak{P}(\mathfrak{A}) \to \mathcal{O}$ is the functor $\mathcal{P}_\Sigma \circ \mathcal{A}$. As for each $i \in I$ the ∞-category $\mathcal{P}_{\Sigma,f}(\mathcal{A}(i))$ is a full subcategory of

$\mathcal{P}_\Sigma(\mathcal{A}(i))$ stable under tensor product and preserved by the transition functors of our diagram I, we also have a $\mathcal{O}$-monoid

$$\mathfrak{P}_f(\mathfrak{A}) \to \mathcal{O}$$

that gives after unstraightening a symmetric monoidal lift of the functor $\mathcal{P}_{\Sigma,f} \circ \mathcal{A}$.

Recall that the Spanier-Whitehead stabilisation functor $\mathrm{SW}(-)$ is given by sending a prestable category $\mathcal{C}$ to the colimit of the diagram

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \cdots .$$

By construction the tensor product on each $\mathcal{P}_{\Sigma,f}(\mathcal{A}(i))$ commutes with finite colimits hence with the suspension functor Σ. This produces a diagram

$$\mathfrak{P}_f(\mathfrak{A}) \xrightarrow{\Sigma} \mathfrak{P}_f(\mathfrak{A}) \xrightarrow{\Sigma} \mathfrak{P}_f(\mathfrak{A}) \xrightarrow{\Sigma} \cdots$$

in the ∞-category $\mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty)$ of $\mathcal{O}$-monoids in Cat_∞ so that by taking the colimit (it exists by [Lur22, Corollary 3.2.3.3]) we obtain an object $\mathfrak{K}$ of $\mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty)$ that, when unstraightened, gives a functor

$$\mathcal{K} \colon I \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

that puts a symmetric monoidal structure on the functor $\mathcal{K}^b \circ \mathcal{A}$.

The ∞-categories $\mathcal{D}^b(\mathcal{A}(i))$ are obtained by inverting quasi-isomorphisms in the ∞-categories $\mathcal{K}^b(\mathcal{A}(i))$. We can now use Hinich's result [Hin16, Proposition 2.1.4] on fiber-wise localisation of cocartesian fibrations to conclude, applied to the marking induced by quasi-isomorphisms on the cocartesian fibration

$$p : \mathfrak{K} \to \mathcal{O}.$$

The hypothesis to apply Hinich's proposition are verified because the tensor product of quasi-isomorphisms is again a quasi-isomorphism, and because the transitions in our diagram of categories preserve quasi-isomorphisms. This gives a symmetric monoidal lift of the functor

$$\mathcal{D}^b \circ \mathcal{A} \colon I \to \mathrm{Cat}_\infty,$$

finishing the proof. The fact that the natural transformation $\mathcal{A} \to \mathcal{D}^b \circ \mathcal{A}$ is symmetric monoidal is easy because it is fully faithful and the tensor product is t-exact in each variable. □

Using the same method as above one can prove:

Proposition 3.7. *Let $\mathcal{C} : I \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ be a diagram of stably symmetric monoidal ∞-categories such that for each $i \in I$ the ∞-category $\mathcal{C}(i)$ has a t-structure such that the tensor product is t-exact in each variable and every arrow $i \to j$ in I induces a t-exact functor. Then the canonical functors $\mathcal{D}^b(\mathcal{C}(i)^\heartsuit) \to \mathcal{C}(i)$ assemble to give a natural transformation $\mathcal{D}(\mathcal{C}^\heartsuit) \Rightarrow \mathcal{C}$ of functors $I \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$.*

Proof. The proof is very similar to the proof of the Proposition 3.6 except that one has to keep track of a $\mathcal{O}$-monoidal functor $\mathfrak{A} \to \mathfrak{C}$ at each time, where $\mathfrak{C} \to \mathcal{O}$ classifies the functor $\mathcal{C}\colon I \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$, and use the successive universal properties of $\mathcal{P}_{\Sigma,f}$, $\mathrm{SW}(-)$ and localisation, that are symmetric monoidal thanks to [Lur22, Proposition 4.8.1.10] and [Hin16, Proposition 2.1.4]. □

We will use the following two theorems :

Theorem 3.8 (Theorem 3.3.1 of [NRS20])**.** *Let $\mathcal{D}$ be an ∞-category which admits finite limits and let $\mathcal{C}$ be an ∞-category. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor which preserves finite limits. Then G admits a left adjoint if and only if $\mathrm{ho}(G) : \mathrm{ho}(\mathcal{D}) \to \mathrm{ho}(\mathcal{C})$ does.*

Corollary 3.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be a stable ∞-categories. Let $G : \mathcal{D} \to \mathcal{C}$ be an exact functor. Then G admits a left (*resp. *a right) adjoint if and only if $\mathrm{ho}(G) : \mathrm{ho}(\mathcal{D}) \to \mathrm{ho}(\mathcal{C})$ does.*

Proof. The case of the left adjoint is the above theorem, and the case of the right adjoint follows by passing to the opposite ∞-categories. □

Theorem 3.10 (Theorem 7.6.10 of [Cis19])**.** *Let $F : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories having finite limits. Assume that F preserves finite limits. Then F is an equivalence of ∞-categories if and only if $\mathrm{ho}(F) : \mathrm{ho}(\mathcal{D}) \to \mathrm{ho}(\mathcal{C})$ is an equivalence of categories.*

To prove descent statement for categories of perverse Nori motives and mixed Hodge modules we will use the following proposition and its convenient corollary.

Proposition 3.11. *Let $\chi : (\mathcal{C}^{\mathrm{op}})^{\triangleleft} \to \mathrm{Cat}_\infty$ be a functor. For any $f : c \to c'$ in $(\mathcal{C}^{\mathrm{op}})^{\triangleleft}$, denote $f^* = \chi(f)$, and assume that any such f^* has a right adjoint f_*. For each $c \in \mathcal{C}$, denote $f_c : c \to v$ the unique map from c to the end point v (we have that $(\mathcal{C}^{\mathrm{op}})^{\triangleleft} \simeq (\mathcal{C}^{\triangleright})^{\mathrm{op}}$.) Let $\overline{\pi} : \overline{\mathcal{D}} \to (\mathcal{C}^{\mathrm{op}})^{\triangleleft}$ be the cocartesian fibration that classifies χ, and let $\pi : \mathcal{D} \to \mathcal{C}^{\mathrm{op}}$ be its pullback by the inclusion $\mathcal{C}^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}})^{\triangleleft}$.*
Then χ is a limit diagram if and only if the following two conditions are verified :

1. *The family $(f_c^*)_{c \in \mathcal{C}}$ is conservative.*
2. *For any cocartesian section $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ of π, the limit $X(v) := \lim_{c \in \mathcal{C}} (f_c)_* X(c)$ exists in $\chi(v)$ and the map $f_c^* X(v) \to X(c)$ adjoint to the canonical map*

$$\lim_c (f_c)_* X(c) \to (f_c)_* X(c)$$

is an equivalence.

Proof. By [Lura, Construction 02T0], there is a diffraction functor

$$\mathrm{Df} : \overline{\mathcal{D}}_v \to \mathrm{Fun}^{\mathrm{Cocart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{D}) \tag{3.11.1}$$

that fits in a commutative triangle :

$$\begin{array}{ccc} \overline{\mathcal{D}}_v & \xrightarrow{\mathrm{Df}} & \mathrm{Fun}^{\mathrm{Cocart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{D}) \\ & {\scriptstyle \mathrm{ev}_v}\nwarrow \quad \nearrow{\scriptstyle \mathrm{res}} & \\ & \mathrm{Fun}^{\mathrm{Cocart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft},\overline{\mathcal{D}}) & \end{array} \quad . \tag{3.11.2}$$

By the diffraction criterion [Lura, Theorem 02T8], χ is a limit diagram if and only if the restriction morphism in (3.11.2) is an equivalence. In that case by [Lura, Remark 02TF], the functor Df is an equivalence.

Assume that for any cocartesian section $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ of π, the functor $c \mapsto (f_c)_*X(c)$ admits a limit in $\overline{D}_v$. By the preservation of limits of any left adjoint in $(\mathcal{C}^{\mathrm{op}})^{\triangleleft}$ and [Lura, Corollary 0311 and Corollary 02KY], this is equivalent to saying that for any cocartesian section $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ of π, the lifting problem

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{op}} & \xrightarrow{X} & \overline{\mathcal{D}} \\ {\scriptstyle i}\big\downarrow & \overset{\overline{X}}{\nearrow} & \big\downarrow{\scriptstyle \overline{\pi}} \\ (\mathcal{C}^{\mathrm{op}})^{\triangleleft} & = & (\mathcal{C}^{\mathrm{op}})^{\triangleleft} \end{array} \tag{3.11.3}$$

admits a solution $\overline{X}$ which is a $\overline{\pi}$-limit ([Lura, Definition 02KG].) By [Lura, Example 02ZA], in that case $\overline{X}$ is the right Kan extension functor along the inclusion $i : \mathcal{C}^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}})^{\triangleleft}$. This easily implies that the restriction functor res : $\mathrm{Fun}^{\mathrm{Cocart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft},\overline{\mathcal{D}}) \to \mathrm{Fun}^{\mathrm{Cocart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{D})$ is fully faithful as the left adjoint of the right Kan extension along $i : \mathcal{C}^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}})^{\triangleleft}$.

Therefore, assuming point 2. of the proposition, χ is a limit diagram if and only if the restriction functor is conservative by [Lura, Corollary 03UZ]. But as the family of evaluations $\mathrm{ev}_c : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{D}) \to \mathcal{D}$ is conservative, this is equivalent to the family of functors $\mathrm{ev}_c \circ \mathrm{res} = \mathrm{ev}_c \circ \mathrm{Df} \circ \mathrm{ev}_v = f_c^*$ being conservative, which is point 1. of the proposition.

By (3.11.2), if χ is a limit diagram, Df is an equivalence which implies that 1. holds. To finish the proof of the proposition, we therefore only have to show that if χ is a limit diagram, then 2. holds. Let $\overline{X}$ be a cocartesian section of $\overline{\pi}$. Assume first that $\overline{X}$ is a $\overline{\pi}$-limit diagram. By [Lura, Example 03ZA] this is equivalent to suppose that $\overline{X}$ is the $\overline{\pi}$-right Kan extension of its restriction X to $\mathcal{C}^{\mathrm{op}}$. Then by [Lura, Corollary 0307], the limit of $c \mapsto (f_c)_*X(c)$ exists and the fact that $\overline{X}$ is cocartesian is equivalent to the fact that for every $c \in \mathcal{C}$, the canonical map

$$f_c^*(\lim_{c'}(f_{c'})_*X(c)) \to X(c) \tag{3.11.4}$$

is an equivalence.

Therefore, to show that 2. holds, we only have to show that any cocartesian section $\overline{X}$ of $\overline{\pi}$ is a $\overline{\pi}$-limit. This is exactly what is proved in the second paragraph of the proof of [Lur11, Theorem 5.17].

□

Corollary 3.12 (See also [Lur22, Corollary 5.2.2.37]). *Let $\chi, \chi' : (\mathcal{C}^{\mathrm{op}})^{\triangleleft} \to \mathrm{Cat}_\infty$ be two functors as in Proposition 3.11, and assume one has a natural transformation $B : \chi \to \chi'$ which commutes with the rights adjoints. Suppose also that $B_v : \chi(v) \to \chi'(v)$ preserves the limits of point 2. in Proposition 3.11 and that each B_c for $c \in \mathcal{C}$ is conservative. Then if χ' is a limit diagram, so is χ.*

3.2 Enhancement of the 6 functors formalism.

In this subsection, we will work with perverse Nori motives or mixed Hodge modules identically. Therefore we denote by Var_k the category of quasi-projective k-schemes when dealing with perverse Nori motives or the category of separated and reduced finite type $\mathbb{C}$-schemes when dealing with mixed Hodge structures. We will call elements of Var_k varieties or k-varieties. For $X \in \mathrm{Var}_k$, we will denote by $\mathcal{M}_{\mathrm{perv}}(X)$ the category of perverse Nori motives constructed in [IM22] or the category of Saito's mixed Hodge modules constructed in [Sai90]. We will have to use realisation functors hence we denote by $\mathrm{rat} : \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathrm{D}^b_c(X)$ either the Betti realisation of perverse Nori motives when $k \subset \mathbb{C}$, the underlying $\mathbb{Q}$-structure functor of mixed Hodge modules, or the ℓ-adic realisation of perverse Nori motives. Here $\mathrm{D}^b_c(X)$ is the derived category of bounded cohomologically constructible sheaves on X^{an} in the two first cases, and the derived category of constructible ℓ-adic étale sheaves on X. The triangulated category $\mathrm{D}^b_c(X)$ is endowed with a perverse and a constructible t-structure for which rat is t-exact and conservative.

We begin with a recollection on the ∞-categorical enhancement of derived categories of constructible sheaves.

Proposition 3.13. *There exists a functor*

$$\mathcal{D}^b_c\colon \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

such that for each map $f : Y \to X$ in Var_k, the symmetric monoidal induced by $\mathcal{D}^b_c(f)$ on the homotopy categories is the classical pullback of constructible sheaves.

Proof. In both analytic sheaves an étale sheaves, for $X \in \mathrm{Var}_k$, the category $\mathrm{D}^b_c(X)$ is embedded in a bigger category $\mathcal{D}(X)$ which is of the form $\mathcal{D}(\mathrm{Shv}(\mathfrak{X}, \mathscr{O}))$ with $\mathfrak{X}$ a topos and $\mathscr{O}$ a sheaf of rings on $\mathfrak{X}$ and is naturally a symmetric monoidal stable ∞-category. Indeed, in the case of analytic sheaves one can take $\mathfrak{X}$ to be the categories of analytic opens of the complex points of X and $\mathscr{O}$ is the constant sheaf $\mathbb{Q}_X$. In the case of ℓ-adic sheaves, following [HRS23] one can take $\mathfrak{X}$ to be the proétale topos of X, and $\mathscr{O}$ to be the sheaf of ring induced by the condensed ring $\mathbb{Q}_\ell = \mathbb{Z}_\ell[1/\ell]$. This construction has a canonical lift to the world of symmetric monoidal ∞-categories thanks to [Lurb, Corollary 2.1.2.3]. As the constructions $X \mapsto \mathfrak{X}$ and $\mathfrak{X} \mapsto \mathcal{D}(\mathrm{Shv}(\mathfrak{X}, \mathscr{O}))$ are functorial, we obtain a functor

$$\mathcal{D} : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

Now as for each f in Var_k, the functor $f^*\colon \mathcal{D}(Y) \to \mathcal{D}(X)$ preserves constructible objects, this induces a functor

$$\mathcal{D}^b_c : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

which gives the usual pullback on homotopy categories by definition in the case of analytic sheaves, and by [HRS23, Theorem 7.7] for ℓ-adic sheaves.

□

Construction 3.14. We have a symmetric monoidal homotopy 2-functor

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-)) : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_1).$$

By restricting to the constructible hearts, this induces a functor

$$\mathcal{M}_{\mathrm{ct}}(-) : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_1)$$

such that the tensor product is exact in each variable and the pullback functors are exact (this can be checked after realisation.) By applying Proposition 3.6 we obtain a functor

$$\mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(-)) : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

Moreover, the equivalence Corollary 2.19 lifts to an equivalence of ∞-categories thanks to Theorem 3.10. The Proposition 3.5 then gives a functor

$$\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(-)) : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty),$$

canonically equivalent to $\mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(-))$.

Proposition 3.15. *Let $f : Y \to X$ be a map in Var_k. Then the functor induced by $\mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(f))$ on the homotopy categories is the pullback functor constructed by Ivorra and Morel or by Saito, depending on which case we are.*

Proof. We claim that if we know that the pullback functors constructed by Ivorra, Morel and Saito admit ∞-categorical lifts f^*, then they will be canonically equivalent to $f^*_{\mathrm{new}} := \mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(f))$. Indeed in that case the ∞-functors f^* and f^*_{new} coincide on the constructible heart by definition, hence they are equivalent by Corollary 3.4.

We prove this first for mixed Hodge modules: For any map $f : Y \to X$ of varieties, there is the factorisation

$$Y \xrightarrow{i} X \times Y \xrightarrow{p} X$$

where i is the closed immersion given by the inclusion of the graph of f (our varieties are separated) and p is the projection. By definition, we have $f^* \simeq p^* \circ i^*$. The functor p^* is given by the external product with the constant object: $p^*(M) = M \boxtimes \mathbb{Q}_Y$, hence is an ∞-functor because the external product is defined as the derived functor on the perverse hearts. The functor i^* is the left adjoint of the functor i_*, which is obtained ([Sai90, (4.2.4) and (4.2.10)]) by deriving the functor i_* on the perverse heart and is therefore an ∞-functor. Thus by Corollary 3.9 the functor i^* is an ∞-functor. Finally f^* admits an ∞-categorical lift.

Now for perverse Nori motives, any map $f : Y \to X$ being a map of quasi-projective varieties, factors as

$$Y \xrightarrow{i} Z \xrightarrow{g} X$$

with i a closed immersion and g a smooth map. Hence $f^* \simeq i^* g^*$. The case of i^* is the same as for mixed Hodge modules, and thus we have an ∞-functor. For the smooth map g, we can assume that X is smooth (as a direct sum of ∞-functors is an ∞-functor), so that g is smooth of pure relative dimension d for some $d \in \mathbb{N}$. Now by definition $f^* = (f^*[d])[-d]$ where $f^*[d]$ is the derived functor of the exact functor $f^*[d]$ on the perverse heart, hence we have an ∞-functor. □

Corollary 3.16. *Let $f : Y \to X$ be a map in $\mathrm{Var}_k^{\mathrm{op}}$, then the ∞-functor*

$$f^* : \mathcal{D}(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}(\mathcal{M}_{\mathrm{perv}}(Y))$$

admits a right adjoint f_*. If f is moreover smooth, f^* admits a left adjoint $f_\sharp$.

Proof. These adjoints exist on the homotopy categories, thus they have ∞-categorical lifts by Corollary 3.9. □

Proposition 3.17. *For each $X \in \mathrm{Var}_k$, the monoidal structure obtained in Construction 3.14 induces on the homotopy categories the same monoidal structure as the one constructed by M. Saito for mixed Hodge modules and by Terenzi ([Ter24]) for perverse Nori motives.*

Moreover, the projection formula for a smooth morphism holds in $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$.

Proof. Note that in both [Ter24] and [Sai90] the tensor product is built in the following way : there is an external tensor product $\boxtimes : \mathcal{M}_{\mathrm{perv}}(X) \times \mathcal{M}_{\mathrm{perv}}(X) \to \mathcal{M}_{\mathrm{perv}}(X \times X)$ that is exact in both variable hence induces by Theorem 3.3 an ∞-functor $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X \times X))$ which we compose with the pullback Δ^* under the diagonal $\Delta : X \to X \times X$ to obtain the tensor product. The other structure morphism of the monoidal structure are obtained similarly. Denote by $\otimes^T : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ Saito's or Terenzi's tensor product and denote by $\otimes^\infty : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ the one we just constructed. We want to check that $\mathrm{ho}(\otimes^\infty) = \mathrm{ho}(\otimes^T)$. For this there are several isomorphisms to check (see [ML98, Section XI.1]) :

1. A functorial isomorphism $\otimes^T \Rightarrow \otimes^\infty$.
2. Modulo point 1. isomorphism, an identification of the unit isomorphism $\rho : M \otimes 1 \to M$ and $\lambda : 1 \otimes M \to M$.
3. Modulo point 1. isomorphism, an identification of the associativity isomorphism $c : M \otimes (N \otimes P) \to (M \otimes N) \otimes P$.

4. Modulo point 1. isomorphism, an identification of the commutativity isomorphism $\gamma : M \otimes N \to N \otimes M$.

By definition, $\otimes^\infty$ coincides with $\otimes^T$ when restricted to $\mathcal{M}_{\text{ct}}(X) \times \mathcal{M}_{\text{ct}}(X)$. By Theorem 3.3 this gives 1. We explain how to obtain 3. and the other checks will be left to the reader as they are very similar. Denote by $t_1^\infty : \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X))$ the functor sending M, N, P to $M \otimes^\infty (N \otimes^\infty P)$ and by $t_2^\infty : \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X)) \times \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\text{perv}}(X))$ the functor sending M, N, P to $(M \otimes^\infty N) \otimes^\infty P$. The associativity isomorphism is a natural equivalence $c^\infty : t_1^\infty \Rightarrow t_2^\infty$. We give the same definition for t_1^T , t_2^T and c^T. Again by definition, the image by the restriction functor of the morphisms c^∞ and c^T are the same, hence up to the equivalence of 1., we obtain 3.

For the projection formula, the arrow exists by a play of adjunctions, and is an equivalence on the homotopy categories. □

Corollary 3.18. *Let X be a variety. The tensor structure on $\mathcal{D}^b(\mathcal{M}_{\text{perv}}(X))$ is closed.*

Proof. Again this is true on the homotopy categories and we can apply Corollary 3.9. □

Proposition 3.19. *Let X be a variety. The realisation functor*

$$R : \mathrm{D}^b(\mathcal{M}_{\text{perv}}(X)) \to \mathrm{D}^b_c(X)$$

admits a canonical lift to a map in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$.

Moreover, it induces a natural transformation

$$\mathcal{D}^b(\mathcal{M}_{\text{perv}}(-)) \Rightarrow \mathcal{D}^b_c(-)$$

on functors

$$\mathrm{Var}_k^{\text{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

Proof. In the case of the Betti realisation this is easy as we have an equivalence

$$\mathcal{D}^b(\mathrm{Cons}(X)) \simeq \mathcal{D}^b_c(X)$$

thanks to Nori's theorem. This enables us to consider the functor $\Delta^1 \times \mathrm{Var}_k^{\text{op}} \to \mathrm{CAlg}(\mathrm{Cat}_1)$ that send $(0, f)$ to $f^* : \mathcal{M}_{\text{ct}}(Y) \to \mathcal{M}_{\text{ct}}(X)$, the edge $(1, f)$ to $f^* : \mathrm{Cons}(Y) \to \mathrm{Cons}(X)$ and $(0 \to 1, X)$ to the realisation. Therefore by Proposition 3.6 we obtain the result.

For the ℓ-adic realisation it is not true that the canonical functor

$$\mathrm{real}_X : \mathcal{D}^b(\mathrm{Cons}(X, \mathbb{Q}_\ell)) \to \mathcal{D}^b_c(X, \mathbb{Q}_\ell)$$

is an equivalence. What we did in the case of the Betti realisation gives a symmetric monoidal natural transformation

$$\mathcal{D}^b(\mathcal{M}_{\text{ct}}(-)) \Rightarrow \mathcal{D}^b(\mathrm{Cons}(-, \mathbb{Q}_\ell)).$$

Hence we have to check that the real_X assemble to give a symmetric monoidal natural transformation

$$\mathcal{D}^b(\mathrm{Cons}(-,\mathbb{Q}_\ell)) \Rightarrow \mathcal{D}^b_c(-).$$

This is Proposition 3.7. □

Corollary 3.20. *The realisation functor*

$$\mathcal{R}\colon \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(-)) \to \mathcal{D}^b_c(-)$$

commutes with all the operations constructed above.

Proof. Indeed, this is true on the homotopy categories, and the exchange morphisms exist because $\mathcal{R}$ commutes with pullbacks and tensor products by definition. □

For free, we also obtain the exceptional functoriality:

Corollary 3.21. *The exceptional functors $f^!$ and $f_!$ constructed by Ivorra, Morel and Saito have ∞-categorical lifts that assemble to give functors*

$$\mathrm{Sch}_k^{(\mathrm{op})} \to \mathrm{Cat}_\infty.$$

Moreover Verdier duality is a natural transformation between the star functoriality (f_* or f^*) and the shriek functoriality of the opposite category ($f_!$ or $f^!$).

Proof. The last statement of this corollary gives a construction. Indeed the formula $\mathbb{D}\circ f_* \simeq f_!\mathbb{D}$ tells us that if we apply Proposition 3.5 to the isomorphisms $\mathbb{D}_X : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))^{\mathrm{op}}$, the obtained functor $\mathrm{Sch}^{(\mathrm{op})} \to \mathrm{Cat}_\infty$ encode exactly the exceptional functoriality. □

3.3 Descent.

Lemma 3.22. *Let $f : Y \to X$ be either a separated étale map or a proper map. Then the cohomological amplitude of the functor f_* on $\mathbb{Q}_\ell$-sheaves and thus perverse Nori motives is less than $2\dim Y + \dim X + 1$.*

Proof. By conservativity of the ℓ-adic realisation, it suffices to deal with $\mathbb{Q}_\ell$-sheaves. If X is a point and for ℓ-torsion sheaves this is Artin's [SGA72, Exposé X Corollaire 4.3]. By proper base change, this gives the result for $f_!$ and ℓ-torsion sheaves. Now, the functor $\mathrm{D}^b(X,\mathbb{Z}_\ell) \to \mathrm{D}^b(X,\mathbb{Z}/\ell\mathbb{Z})$ is conservative and of cohomological amplitude 1 (the short exact sequence $0 \to \mathbb{Z}_\ell \xrightarrow{\times\ell} \mathbb{Z}_\ell \to \mathbb{Z}/\ell\mathbb{Z} \to 0$ gives that $\mathrm{H}^{-1}(\mathrm{H}^0(M)\otimes\mathbb{F}_\ell) = \mathrm{H}^0(M)[\ell]$, $\mathrm{H}^0(\mathrm{H}^0(M)\otimes\mathbb{F}_\ell) = \mathrm{H}^0(M)/\ell$ and $\mathrm{H}^i(\mathrm{H}^0(M)\otimes\mathbb{F}_\ell) = 0$ for $i > 0$) , and tensoring by $\mathbb{Q}_\ell$ at most makes the amplitude be smaller. Hence this gives the cohomological amplitude for $f_!$ on $\mathbb{Q}_\ell$ sheaves.

If f is proper, $f_! = f_*$ and we have the lemma. If f is étale, we can factor f as an open immersion in a finite étale morphism. The finite morphism is dealt with the proper case, and the pushforward under an open immersion is t-exact. □

Theorem 3.23. *The functor $X \mapsto \mathcal{D}^b_{\mathcal{M}}(X)$ is a sheaf with respect to the étale and h topologies.*

Proof. We first deal with the case of perverse Nori motives: The functor $X \mapsto \mathcal{D}^b_c(X, \mathbb{Q}_\ell)$ is isomorphic to the functor $\mathcal{D}_{\text{cons}}(-, \mathbb{Q}_\ell)$ of [HRS23, Definition 3.3] by [HRS23, Theorem 3.45]. The functor $\mathcal{D}_{\text{cons}}(-, \mathbb{Q}_\ell)$ is an étale sheaf by [HRS23, Theorem 3.13]. By [CD19, Corollary 3.3.38], it is also a h-sheaf because of localisation and proper base change (see also [RS20, Theorem 2.1.15]).

Moreover, the ℓ-adic realisation $\mathcal{D}^b(\mathcal{M}_{\text{ct}}) \to \mathcal{D}_{\text{cons}}(-, \mathbb{Q}_\ell)$ is conservative. By Corollary 3.12 one just have to check that it preserves the limits in the Proposition 3.11 for $\mathcal{C}$ the Cech nerve of a morphism of schemes $f : Y \to X$. Let F be a cocartesian section as in the proposition, that is, for each $n \in \mathbb{N}$,

$$M_n := F(\underbrace{Y \times_X \cdots Y}_{n+1\text{-times}}) \in \mathcal{D}^b(\mathcal{M}_{\text{ct}}(X))$$

which verify in particular that for each p_n one of the projections $\underbrace{Y \times_X \cdots Y}_{n+2\text{-times}} \to \underbrace{Y \times_X \cdots Y}_{n+1\text{-times}}$, we have $p_n^* M_n \simeq M_{n+1}$. Let $a \in \mathbb{N}$ be an integer such that $M_0 \in \mathcal{D}^{[-a,a]}(\mathcal{M}_{\text{ct}}(Y))$. Then as each pullback is t-exact, each M_n is also in $\mathcal{D}^{[-a,a]}(\mathcal{M}_{\text{ct}}(Y^{n+1}))$. By Lemma 3.22 the cohomological dimension of $f_* : \mathcal{D}^b_c(Y, \mathbb{Q}_\ell) \to \mathcal{D}^b_c(X, \mathbb{Q}_\ell)$ is $\dim X + 2 \dim Y + 1$ for $f : Y \to X$ separated étale morphism of finite type k-schemes. Hence if $f_n : \underbrace{Y \times_X \cdots Y}_{n+1\text{-times}} \to X$ is the obvious map, it is étale by base change and hence has ℓ-adic cohomological dimension $d = 3 \dim X + 1$. Therefore we have $(f_n)_* M_n \in \mathcal{D}^{[-a-d,a+d]}(\mathcal{M}_{\text{ct}}(X))$. But $\mathcal{D}^{[-a-d,a+d]}(\mathcal{M}_{\text{ct}}(X))$ and $\mathcal{D}^{[-a-d,a+d]}_{\text{cons}}(X, \mathbb{Q}_\ell)$ are (2(a+d)+1)-categories, in which totalisations (simplicial limits) are finite limits so the realisation commutes with them.

For mixed Hodge modules, one has to do the same thing but with the forgetful functor to constructible sheaves, which is an étale sheaf (note that in the analytic side, an étale morphism is a local isomorphism.) □

4 The realisation functor and applications.

4.1 Construction of the realisation.

In this section, we construct realisation functors from the category of étale motives to the derived categories of mixed Hodge modules and perverse Nori motives that commute with the 6 operations. We use Drew and Gallauer' result in [DG22] in which they prove that the stable motivic $\mathbb{A}^1$-invariant ∞-category $\mathcal{SH}$ affords a universal property among systems of ∞-categories called *coefficient systems*: these are functors

$$\text{Sch}_k^{\text{op}} \to \text{CAlg}(\text{Cat}_\infty)$$

that have the minimal functoriality needed for the 6 operations to exist thanks to Ayoub's thesis (see Appendix A for a precise definition and some results on coefficient systems.) Any coefficient system $C \in \mathrm{CoSys}_k$ which is cocomplete (this means that the underlying functor $C\colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Cat}_\infty$ takes values in symmetric monoidal cocomplete ∞-categories) receives an unique realisation

$$\mathcal{SH} \to C$$

that commutes with enough operations to ensures that it commutes with the 6 operations when restricted to a suitable subcategory of constructible objects. Hence we show that our system of categories $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(-))$ and $\mathcal{D}^b(\mathrm{MHM}(-))$ are coefficient systems, so that by taking Ind-objects we obtain cocomplete coefficient systems.

Notation 4.1. *For X a separated non reduced finite type k-scheme we define*

$$\mathcal{D}^b(\mathrm{MHM}(X)) := \mathcal{D}^b(\mathrm{MHM}(X_{\mathrm{red}})).$$

By localisation for mixed Hodge modules this is well defined. We also denote by $\mathcal{D}^b_{\mathcal{M}}$ the functor $\mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ of Construction 3.14.

Theorem 4.2. *The functor $\mathcal{D}^b_{\mathcal{M}}$ defines an object of $\mathrm{CoSys}_k^{\text{ét}}$.*

Proof. This is the content of Section 3.2, together with Theorem 3.23 and Proposition A.5. See Definition A.1 for the precise definition of coefficient systems.

□

In Proposition A.5 we prove that coefficients systems, defined as a class of functors

$$\mathrm{QProj}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

on quasi-projective k-varieties, are equivalent to coefficients systems

$$\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

with source the category of every finite type k-scheme, when they afford étale descent. The enables us to extend perverse Nori motives and mixed Hodge modules to every finite type k-scheme, together with all the operations.

Remark 4.3. Using that the Hom in $\mathcal{D}^b_{\mathcal{M}}$ and the Ext^i in $\mathcal{M}_{\mathrm{perv}}$ and $\mathcal{M}_{\mathrm{ct}}$ are étale sheaves (for the perverse heart, one has to use affine étale morphisms), one can show that after extension to finite type k-schemes, we still have

$$\mathcal{D}^b_{\mathcal{M}}(X) \simeq \mathcal{D}^b(\mathcal{M}_{\mathrm{ct}}(X)) \simeq \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$$

and in the case of perverse Nori motives, the category $\mathcal{M}_{\mathrm{perv}}(X)$ is the universal category as for quasi-projective schemes. The weight structures also extend to that case.

Theorem 4.4. *There exists an (essentially unique) morphism of coefficients systems $\mathcal{DM} \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ on the category Sch_k. It restricts to a functor*

$$\nu : \mathcal{DM}_c \to \mathcal{D}^b_{\mathcal{M}}. \tag{4.4.1}$$

The functor (4.4.1) commutes with the 6 operations, that is all the exchange transformations are isomorphisms.

Proof. Thanks to Theorem 4.2 we know that $\mathcal{D}^b_{\mathcal{M}}$ is an object of CoSys_k, hence its indization $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ is an object of $\mathrm{CoSys}^{\mathrm{c}}_k$. Together with Theorem A.2, this gives a morphism of coefficient systems

$$\mathcal{SH} \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}.$$

As $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ is $\mathbb{Q}$-linear, this morphism of coefficient systems factors through $\mathcal{SH} \to \mathcal{SH}_{\mathbb{Q}} = \mathcal{DM}_{\mathrm{Nis}}$ to give a morphism of coefficient systems

$$\mathcal{DM}_{\mathrm{Nis}} \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}.$$

Note that this is in fact a morphism of pullback formalism in the sense of [DG22, Definition 2.2], and as $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ satisfies non effective étale hyperdescent (indeed because the mapping spectra $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ are in $\mathcal{D}(\mathbb{Q})$ the [CD19, Theorem 3.3.23] implies that this is a consequence of non effective étale descent which is verified because indization of a fully faithful functor is fully faithful), this map factors through the étale hyper-sheafification of $\mathcal{DM}_{\mathrm{Nis}}$, which is $\mathcal{DM}$, to give a morphism of coefficient systems

$$\tilde{\nu} : \mathcal{DM} \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}.$$

Denote by $\mathcal{DM}_c(X)$ (*resp.* by $\mathcal{D}^{\mathrm{gm}}_{\mathcal{M}}(X)$) the thick full subcategory of $\mathcal{DM}(X)$ (*resp.* of $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}(X)$) spanned by the $f_\sharp\mathbb{Q}_Y(i)$ for $f : Y \to X$ smooth and $i \in \mathbb{Z}$. As $\mathcal{D}^b_{\mathcal{M}}(X) \subset \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}(X)$ is thick and each $f_\sharp\mathbb{Q}_Y(i)$ is in $\mathcal{D}^b_{\mathcal{M}}(X)$, we have $\mathcal{D}^{\mathrm{gm}}_{\mathcal{M}}(X) \subset \mathcal{D}^b_{\mathcal{M}}(X)$.

Moreover, $\tilde{\nu}$ commutes with all the f^*, $f_\sharp$ and is monoidal. Note that the Tate twist $M(i) = M \otimes \mathbb{Q}_X(1)$ is tensorisation by $\mathbb{Q}_X(1) \simeq \mathrm{fib}(p_\sharp\mathbb{Q}_{\mathbb{P}^1_X} \to \mathbb{Q}_X)[-2]$ with $p : \mathbb{P}^1_X \to X$ and the map $p_\sharp\mathbb{Q}_{\mathbb{P}^1_X} \to \mathbb{Q}_X$ is induced by the inclusion $\infty_X \to \mathbb{P}^1_X$. Therefore, $\tilde{\nu}$ also commutes with Tate twists, and induces a morphism of coefficient systems

$$\nu^{\mathrm{gm}} : \mathcal{DM}_c \to \mathcal{D}^{\mathrm{gm}}_{\mathcal{M}} \tag{4.4.2}$$

In particular, at the level of triangulated categories, we obtain a premotivic morphism that verifies all the hypothesis of [CD19, Theorem 4.4.25] : $\mathrm{D}^{\mathrm{gm}}_{\mathcal{M}}$ is τ-generated for the Tate twist, $\mathcal{DM}_c$ is τ-dualisable by absolute purity, $\mathrm{D}^{\mathrm{gm}}_{\mathcal{M}}$ is separated because it is a sub pullback formalism of $\mathcal{D}^b_{\mathcal{M}}$ which satisfy étale descent by Theorem 3.23 (which is equivalent to separateness by [CD19, Corollaire 3.3.34] in the $\mathbb{Q}$-linear case) and all Tate twist are invertible in $\mathcal{DM}_c$ by construction.

Therefore, the functor ν^{gm} commutes with the 6 operations. One has to be a little careful here: a priori, [CD19, Theorem 4.4.25] implies only that ν^{gm} commutes with all the operations when $\mathcal{D}^{\mathrm{gm}}_{\mathcal{M}}$ is endowed with the operations they construct in their book, and not necessarily with the constructions of Saito, Ivorra, Morel as in Corollary 3.21. We know that for $f : Y \to X$ a morphism, f^* is the same for both constructions, hence by adjunction this is also the case for f_*. For $j : U \to X$ an open immersion the $j_\sharp$ are the same, but $j_\sharp = j_!$ hence by Nagata compactification (and the fact that for a proper p, $p_* = p_!$) and composition both $f_!$ are isomorphic for any $f : Y \to X$ separated. By adjunction we also have an isomorphism between the $f^!$. This defines a isomorphism of étale sheaves of ∞-categories, hence we also have compatibility of operations for non separated morphisms (when they are defined for $\mathcal{DM}$). The tensor product is the same, hence also the internal $\mathscr{H}om$ by adjunction.

We have obtained that the functor

$$\nu : \mathcal{DM}_c \to \mathcal{D}^b_{\mathcal{M}} \tag{4.4.3}$$

commute with the 6 operations at the triangulated categories level. But all exchange morphisms exist at the level of ∞-categories, and the functor from an ∞-category to its homotopy category is conservative, hence ν commutes with all the operations. □

Remark 4.5. As noted by Frédéric Déglise, for the construction of the above realisation one does not actually need to prove directly étale descent as in Theorem 3.23 (which requires a somehow complicated argument in order to ensure that the limits of the sheaf condition are finite). Indeed as $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ is a cocomplete coefficient system, as above there is a realisation

$$\mathcal{SH} \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$$

which factors through $\mathcal{SH}_{\mathbb{Q}}$ because $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ is a system of $\mathcal{D}(\mathbb{Q})$-modules. But now by [CD19, Section 16.2] there is a canonical decomposition

$$\mathcal{SH}_{\mathbb{Q}} \simeq \mathcal{SH}^+_{\mathbb{Q}} \times \mathcal{SH}^-_{\mathbb{Q}}$$

with the $+$-part being isomorphic to $\mathcal{DM}(-,\mathbb{Q})$ by [CD19, Theorem 16.2.18]. Therefore to obtain the factorisation of

$$\mathcal{SH}_{\mathbb{Q}} \to \mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$$

through $\mathcal{DM}(-,\mathbb{Q})$ it suffices to show that the image of $\mathbb{Q}^-$ (which is the image of $\mathbb{Q}$ under the projection $\mathcal{SH}_{\mathbb{Q}} \to \mathcal{SH}^-_{\mathbb{Q}}$) in $\mathrm{Ind}\mathcal{D}^b_{\mathcal{M}}$ is zero. But now as the Betti (or ℓ-adic) realisation is conservative on compact objects, this can be check in $\mathcal{D}^b_c$, hence étale locally where it is true because $\mathbb{Q}^-$ vanishes whenever -1 is a sum of squares in the residual fields of the base scheme.

Corollary 4.6. *There exists a Hodge realisation of $R_H : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}) \to \mathcal{D}^b(\mathrm{MHM})$ compatible with the six operations. It factors the Hodge realisation that we have constructed.*

Proof. By Theorem 4.4, there exists a Hodge realisation from $\mathcal{DM}_c(-,\mathbb{Q})$ to the derived category of mixed Hodge modules. Therefore, taking the underlying $\mathbb{Q}$-structure and then perverse ${}^{\mathrm{p}}\mathrm{H}^0$, we obtain a factorisation of the perverse ${}^{\mathrm{p}}\mathrm{H}^0$ of the Betti realisation through $\mathrm{MHM}(X)$. The universal property of perverse Nori motives then gives a faithful exact functor $\mathcal{M}_{\mathrm{perv}}(X) \to \mathrm{MHM}(X)$, which induces a functor $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathrm{MHM}(X))$. Now all the constructions of [IM22] are the constructions of the operations for mixed Hodge modules, hence this functor is compatible with the operations. The fact that this functor factors the Hodge realisation we constructed above follows from the universal property of $\mathcal{DM}_c$. □

Recall that by [IM22, Corollary 6.18], [Bon11, Proposition 2.3.1] and [Hé11, Theorem 3.3] the categories $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$, $\mathcal{D}^b(\mathrm{MHM}(X))$ and $\mathcal{DM}_c(X)$ admit weight structures (for the last one, note that Beilinson motives and étale motives are equivalent by [CD19, Theorem 16.1.2].)

Proposition 4.7. *Let X be a finite type k-scheme. Then the functors*

$$R_H : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathrm{MHM}(X)),$$

$$\nu : \mathcal{DM}_c(X) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$$

and

$$\mathrm{Hdg}^* : \mathcal{DM}_c(X) \to \mathcal{D}^b(\mathrm{MHM}(X))$$

are weight-exact.

Proof. First note that by definition of the weights on $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$, the ℓ-adic realisation $R_\ell : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b_c(X, \mathbb{Q}_\ell)$ is weight exact and reflects weights. As all weight filtrations and structures considered are bounded, it suffices to show that our functors map zero weight objects to zero weight objects. Let us first consider functors with source $\mathcal{DM}_c(X)$. For those, is suffices to check that the image of the relative motive of a smooth proper X-scheme is pure of weight 0. For the ℓ-adic realisation this is [Del80] and [Mor19]. Therefore, the functor $\nu : \mathcal{DM}_c(X) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ is also weight exact. The fact that Hdg^* is exact is the fact that the mixed Hodge modules associated to a smooth proper X-scheme is pure of weight 0, see [Sai06, Theorem 6.7 and Proposition 6.6].

Now is only left $R_H : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{D}^b(\mathrm{MHM}(X))$. By [IM22, Corollary 6.27], any pure complex of perverse Nori motives is the direct sum of shifts of its ${}^{\mathrm{p}}\mathrm{H}^i$, hence to check that R_H is weight exact it suffices to show that it sends pure objects of $\mathcal{M}_{\mathrm{perv}}(X)$ to pure objects of $\mathrm{MHM}(X)$. By construction of R_H, we have a commutative diagram :

$$\mathcal{DM}_c(X) \xrightarrow{\mathrm{H}_u} \mathcal{M}_{\mathrm{perv}}(X) \xrightarrow{R_H} \mathrm{MHM}(X) \quad (\text{with } \mathrm{Hdg}^* : \mathcal{DM}_c(X) \to \mathrm{MHM}(X)) . \tag{4.7.1}$$

With H_u the universal functor defining $\mathcal{M}_{\mathrm{perv}}(X)$. But by [IM22, Corollary 6.27], H_u sends pure objects to pure objects, and any motive $M \in \mathcal{M}_{\mathrm{perv}}(X)$ is a quotient of some $\mathrm{H}_u(N)$ for $N \in \mathcal{DM}_c(X)$. Assume that M is pure of some weight w. Then M is a direct factor of a quotient of the weight filtration of $\mathrm{H}^0(N)$. As Hdg^* is weight exact, it follows that $R_H(M)$ is a direct factor of a weight w mixed Hodge module, hence R_H is weight exact. □

Remark 4.8. Our construction of the realisation functor $\mathcal{DM}_c(X)$ is very similar to that of Ivorra in [Ivo16] (which works for smooth varieties). Indeed, given a smooth variety X, in both definitions one first gives the image of the smooth X-schemes in

$\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$. To be more precise, Ivorra only gives the image of smooth affine X-schemes and then extend to smooth X-schemes by colimits but as the functor $(f : Y \to X) \mapsto f_! f^! \mathbb{Q}_X$ is a Nisnevich cosheaf, its value on smooth X-schemes is indeed determined by its value on smooth affine X-schemes. Once one has the value of the functor on smooth X-schemes, the rest of the construction is just applying the universal property of $\mathcal{DM}_c(X)$: one has to check that the functor on smooth X-schemes is an $\mathbb{A}^1$-invariant étale sheaf and $\mathbb{P}^1$-stable. Therefore, it is easy to see that if both definitions on smooth affine schemes (ours with the 6-operations and Ivorra's using cellular complexes and Beilinson basic lemma) coincide, then our functor $\mathcal{DM}_c(X) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ and Ivorra's are equivalent. In particular, this would prove that Ivorra's construction is compatible with the Betti realisation constructed by Ayoub on Voevodsky motives, hence that the category of perverse Nori motives as in [IM22] is equivalent as the diagram category of relative pairs studied in [Ivo17] and [IM22, Section 2.10].

This can be reduced to check a simple fact, that the author did not manage to do: in [Ivo16, Proposition 4.15] Ivorra proves that for a given smooth affine map $f : Y \to X$ with X a smooth variety, there is a specific cellular stratification $Y^\bullet$ of Y such that the cellular complex induced on Y is an $\mathrm{H}^0 f_!$-acyclic resolution of $f^! \mathbb{Q}_X$, hence that there is an isomorphism in $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ between Ivorra's functor at Y and the homology of Y, that is $\mathrm{ra}_X^{\mathcal{M}_{\mathrm{perv}}}(Y, Y_\bullet) \simeq f_! f^! \mathbb{Q}_X$. This isomorphism induces a canonical and functorial isomorphism on each ${}^{\mathrm{p}}\mathrm{H}^i$. The question is whether one can construct such an isomorphism in a functorial way, as this would give the wanted isomorphism of functors. This does not seem easy, and it is unlikely that one can arrange those specific stratification so that they are compatible with a give morphism of smooth affine X-schemes.

4.2 Relation with the t-structure conjecture.

Let $H : \mathcal{T} \to \mathcal{A}$ be a homological functor from a triangulated category $\mathcal{T}$ to an abelian category $\mathcal{A}$. Let $H_u : \mathcal{T} \to \mathcal{M}_u$ an homological functor and $F_u : \mathcal{M}_u \to \mathcal{A}$ a faithful exact functor such that $H = F_u \circ H_u$. We consider the following universal property :

For all factorisation

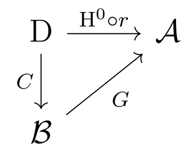
$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{H} & \mathcal{A} \\ {\scriptstyle C}\downarrow & \nearrow_{G} & \\ \mathcal{B} & & \end{array} \tag{4.8.1}$$

with C a homological functor and G a faithful exact functor there exists a unique (up to unique natural isomorphism) faithful exact functor $G_u : \mathcal{M}_u \to \mathcal{B}$ such that $G = F_u \circ G_u$ and $C = G_u \circ H_u$.

By [IM22, Section 1], such an universal factorisation $H = F_u \circ H_u$ always exists, and perverse Nori motives $\mathcal{M}_{\mathrm{perv}}(X)$ are defined as the universal factorisation of the perverse ${}^{\mathrm{p}}\mathrm{H}^0$ of the Betti realisation of Voevodsky motives $\mathcal{DM}_c(X)$.

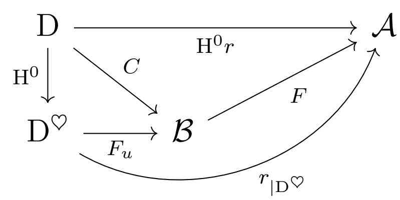
Lemma 4.9. *Let* D *be a triangulated category with a t-structure, and let $r : \mathcal{D} \to \mathrm{D}^b(\mathcal{A})$ be a conservative triangulated t-exact functor. Then the pair* $(\mathrm{H}^0 : \mathrm{D} \to \mathrm{D}^\heartsuit, r_{|\mathrm{D}^\heartsuit} : \mathrm{D}^\heartsuit \to \mathcal{A})$ *is the universal factorisation of* $\mathrm{H}^0 r$ *in the above sense.*

Proof. Indeed, let



be a factorisation of $\mathrm{H}^0 \circ r$ with C homological and G faithful exact.

Then if there is a commutative diagram



,

we have, denoting by $i : \mathrm{D}^\heartsuit \to \mathrm{D}$ the inclusion, because $\mathrm{H}^0 \circ i = \mathrm{Id}$, the isomorphism $F_u = C \circ i$. Now as $F_u \circ F = r_{|\mathrm{D}^\heartsuit}$, the functor F_u is faithful exact, hence $\mathrm{D}^\heartsuit$ affords the universal property. □

Recall the following conjecture ([And04, Section 21.1.7] and [Bei12]):

Conjecture 4.10. *There exits a non degenerate t-structure on $\mathcal{DM}_c(k)$ compatible with the tensor product and such that the ℓ-adic realisation is t-exact.*

In [Bei12] Beilinson proves that if the characteristic of k is zero the conjecture implies that the realisation of $\mathcal{DM}_c(k)$ are conservative. In [Bon15, Theorem 3.1.4], Bondarko proves that if such a t-structure exists for all fields of characteristic 0, then there exists a perverse t-structure on each $\mathcal{DM}_c(X)$ for all X finite type k-schemes. The joint conservativity of the family of x^* for $x \in X$ and of the realisation over fields implies that in that case, the ℓ-adic realisation is conservative and t-exact, when the target is endowed with the perverse t-structure.

Theorem 4.11. *Assume Conjecture 4.10 for all fields of characteristic* 0*. Let X be a finite type k scheme for such a field k. Then the heart of the perverse t-structure of $\mathcal{DM}_c(X)$ is canonically equivalent to $\mathcal{M}_{\mathrm{perv}}(X)$ the category of perverse Nori motives. Moreover, the functor $\nu : \mathcal{DM}_c(X) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ is an equivalence of stable ∞-categories. This implies that $\mathcal{DM}_c(X)$ is both the derived category of its perverse and constructible hearts.*

Lemma 4.12. *Denote by $\mathcal{M}(X)$ the heart of the perverse t-structure on $\mathcal{DM}_c(X)$. There is a canonical equivalence $F_X : \mathcal{M}_{\mathrm{perv}}(X) \simeq \mathcal{M}(X)$. Denote by*

$$\gamma_X : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{DM}_c(X)$$

the canonical functor induced by the inclusion of the heart. The composition $\nu_X \circ \gamma_X$ is equivalent to the identity functor $\mathrm{Id}_{\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))}$.

Proof. By construction and with Lemma 4.9, with the conservativity of the ℓ-adic realisation it is immediate that there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{DM}_c(X) & & \\ {\scriptstyle H_u}\downarrow & \searrow^{\mathrm{H}^0} & \\ \mathcal{M}_{\mathrm{perv}}(X) & \xrightarrow[F_X]{} & \mathcal{M}(X) \end{array} \tag{4.12.1}$$

with H_u the universal homological functor of the definition of perverse Nori motives and F_X an exact equivalence of categories, compatible with the realisation. We identify via F_X the categories $\mathcal{M}_{\mathrm{perv}}(X)$ and $\mathcal{M}(X)$. By the universal property of the bounded derived category Corollary 3.4 we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) & \xrightarrow[\gamma_X]{} & \mathcal{DM}_c(X) & \xrightarrow[\nu_X]{} & \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \\ {\scriptstyle \mathrm{H}^0}\downarrow & & {\scriptstyle \mathrm{H}^0}\downarrow & & {\scriptstyle \mathrm{H}^0}\downarrow \\ \mathcal{M}_{\mathrm{perv}}(X) & \xrightarrow[\mathrm{Id}]{} & \mathcal{M}_{\mathrm{perv}}(X) & \xrightarrow[H^0\nu_X]{} & \mathcal{M}_{\mathrm{perv}}(X) \end{array} \tag{4.12.2}$$

compatible with the realisation. Applying Lemma 4.9 to the derived category of perverse motives $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ we obtain that $\mathrm{H}^0\nu = \mathrm{Id}$. The functor $\nu_X \circ \gamma_X$ is determined by its restriction to the heart $\mathcal{M}_{\mathrm{perv}}(X)$, hence it is the identity. □

We now will check that the functor $\gamma_X \circ \nu_X : \mathcal{DM}_c(X) \to \mathcal{DM}_c(X)$ is also the identity (up to a unique invertible natural transformation.) By [Rob15, Corollary 2.39] it suffices to show that the composition of $\gamma_X \circ \nu_X$ with the Yoneda functor

$$\mathrm{Sm}_X \to \mathcal{DM}_c(X)$$

is the Yoneda functor. For this, it suffices to show that $\gamma_X \circ \nu_X$ commutes with $f_\sharp$ for smooth f, with all pullback functors and with the tensor product. This is true for the functor $\mathcal{DM}_c(X) \to \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$, hence it suffices to prove that the functor

$$\gamma_X : \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathcal{DM}_c(X)$$

commutes with those operations. Now the proof is the similar as the proof of Proposition 3.15: the functor real commutes with functors which are t-exact (up to a shift) for the perverse t-structure, hence with f^* for smooth f, with i_* for a closed immersion, and with the external tensor product. As there is a commutative diagram

$$\begin{array}{ccc} \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) & \longrightarrow & \mathcal{DM}_c(X) \\ & \searrow & \downarrow \\ & & \mathcal{D}^b_c(X, \mathbb{Q}) \end{array}$$

with the vertical functor conservative an commuting with the 6-operations, the functor real also commutes with i^* and f_*. This implies that it commutes with the tensor product and the $f_\sharp$ for smooth f, finishing the proof of Theorem 4.11.

4.3 Nori motives as modules in étale motives.

Notation 4.13. *To be coherent with the notations of [Ayo22], we will denote by* Nor^* *the realisation functor*

$$\mathcal{DM} \to \mathcal{DN}$$

where we have denoted by $\mathcal{DN}$ *the indization of the bounded derived category of perverse motives. We will denote by* $\mathcal{DN}_c$ *its full subcategory of compact objects.*

By construction, the functor Nor^* preserves colimits, hence it has a right adjoint $\mathrm{Nor}_* : \mathcal{DN} \to \mathcal{DM}$. Denote by

$$\mathscr{N}_k = \mathrm{Nor}_*\mathrm{Nor}^*\mathbb{Q}_{\mathrm{Spec}\ k} \in \mathcal{DM}(\mathrm{Spec}\ k, \mathbb{Q}). \tag{4.13.1}$$

This is a $\mathbb{E}_\infty$-ring object in $\mathcal{DM}(X)$. As in Ayoub's [Ayo22, Construction 1.91], Nor^* factors canonically through the functor

$$\mathrm{Mod}_{\mathscr{N}}(\mathcal{DM}) : X \mapsto \mathrm{Mod}_{\mathscr{N}_{|X}}(\mathcal{DM}(X, \mathbb{Q})) \tag{4.13.2}$$

as

$$\mathrm{Nor}^* : \mathcal{DM} \overset{\mathscr{N}\otimes -}{\to} \mathrm{Mod}_{\mathscr{N}}(\mathcal{DM}) \overset{\widetilde{\mathrm{Nor}^*}}{\to} \mathcal{DN}. \tag{4.13.3}$$

Lemma 4.14. *Let k be a field of characteristic zero and let $\pi_X : X \to \mathrm{Spec}\ k$ be a finite type k-scheme. The functor*

$$\widetilde{\mathrm{Nor}^*} : \mathrm{Mod}_{\pi_X^*\mathscr{N}_k}(\mathcal{DM}(X)) \to \mathcal{DN}(X)$$

is fully faithful. Moreover he natural map $\pi_X^*\mathrm{Nor}_*\mathbb{Q}_{\mathrm{Spec}\ k} \to \mathrm{Nor}_*\mathbb{Q}_X$ is an equivalence.

Proof. The proof adds nothing new to [Ayo22, Remark 1.97]. Denote by $\widetilde{\mathrm{Nor}_*}$ the right adjoint of $\widetilde{\mathrm{Nor}^*}$. Assume first that $X = \mathrm{Spec}\ k$. We have to show that the unit map

$$\mathrm{Id} \to \widetilde{\mathrm{Nor}_*}\widetilde{\mathrm{Nor}^*}$$

is an equivalence. By [Iwa18, Lemma 2.6], the ∞-category $\mathrm{Mor}_{\mathscr{N}_k}(\mathcal{DM}(k))$ is compactly generated by the $f_\sharp\mathbb{Q}_Y(i) \otimes \mathscr{N}_k$ for $f : Y \to \mathrm{Spec}\ k$ smooth and $i \in \mathbb{Z}$. As $\widetilde{\mathrm{Nor}^*}$ of those compact generators are compact objects of $\mathcal{DN}(k)$, the right adjoint $\widetilde{\mathrm{Nor}_*}$ preserves colimits. Thus it suffices to check that for any compact object $C \otimes \mathscr{N}_k$ of the above form, the map

$$C \otimes \mathscr{N}_k \to \widetilde{\mathrm{Nor}_*}\widetilde{\mathrm{Nor}^*}(C \otimes \mathscr{N}_k) \simeq \widetilde{\mathrm{Nor}_*}\mathrm{Nor}^*C$$

is an equivalence. As the forgetful functor $\mathrm{Mor}_{\mathscr{N}_k}(\mathcal{DM}(k)) \to \mathcal{DM}(k)$ is conservative it is even sufficient to prove that

$$C \otimes \mathscr{N}_k \to \mathrm{Nor}_* \mathrm{Nor}^* C$$

is an equivalence in $\mathcal{DM}(k)$. This is true because over a field, compact objects in $\mathcal{DM}(k)$ are dualisable and for dualisable objects this map is an equivalence by [Ayo14b, Lemma 2.8].

Now for a general X we use that the functor $\widetilde{\mathrm{Nor}^*}$ commutes with the 6 operations. For compact $M, N \in \mathrm{Mod}_{\pi_X^* \mathscr{N}_k}(\mathcal{DM}(X))$ we have:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Mod}_{\pi_X^*\mathscr{N}_k}(\mathcal{DM}(X))}(M, N) &\simeq \mathrm{Map}_{\mathrm{Mod}_{\pi_X^*\mathscr{N}_k}(\mathcal{DM}(X))}(\mathbb{Q}_X, \mathscr{H}om(M, N)) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathscr{N}_k}(\mathcal{DM}(k))}(\mathbb{Q}_k, (\pi_X)_* \mathscr{H}om(M, N)) \\
&\simeq \mathrm{Map}_{\mathcal{DN}(k)}(\mathbb{Q}_k, \widetilde{\mathrm{Nor}^*}(\pi_X)_* \mathscr{H}om(M, N)) \\
&\simeq \mathrm{Map}_{\mathcal{DN}(k)}(\mathbb{Q}_k, (\pi_X)_* \mathscr{H}om(\widetilde{\mathrm{Nor}^*} M, \widetilde{\mathrm{Nor}^*} N)) \\
&\simeq \mathrm{Map}_{\mathcal{DN}(X)}(\widetilde{\mathrm{Nor}^*} M, \widetilde{\mathrm{Nor}^*} N)
\end{aligned}$$

which finishes the proof of fully faithfulness. Now that we know that the unit

$$\mathrm{Id} \to \widetilde{\mathrm{Nor}_*}\widetilde{\mathrm{Nor}^*}$$

is an equivalence in $\mathrm{Mod}_{\pi_X^*\mathscr{N}_k}(\mathcal{DM}(X))$, by plugging in $\pi_X^*\mathscr{N}_k$ we obtain

$$\pi_X^*\mathscr{N}_k \xrightarrow{\sim} \widetilde{\mathrm{Nor}_*}\widetilde{\mathrm{Nor}^*}(\pi_X^*\mathscr{N}_k)$$

and we have

$$\widetilde{\mathrm{Nor}^*}(\pi_X^*\mathscr{N}_k) \simeq \pi_X^*\widetilde{\mathrm{Nor}^*}(\mathscr{N}_k) \simeq \mathbb{Q}_X.$$

Therefore the canonical map

$$\pi_X^*\mathscr{N}_k \to \widetilde{\mathrm{Nor}_*}\mathbb{Q}_X$$

is an equivalence, thus it stays an equivalence after applying the forgetful functor, leaving us with the equivalence

$$\pi_X^*\mathscr{N}_k \xrightarrow{\sim} \mathrm{Nor}_*\mathbb{Q}_X$$

in $\mathcal{DM}(X)$.

□

From the previous lemma there is no more ambiguity for $\mathscr{N}_X = \pi_X^*\mathscr{N}_k \simeq \mathrm{Nor}_*\mathbb{Q}_X \in \mathcal{DM}(X)$.

Proposition 4.15. *Let $k = \mathrm{colim}_i k_i$ be a filtered colimit of fields of characteristic zero. Then the natural functors*

$$\mathrm{colim}_i \mathcal{DN}_c(\mathrm{Spec}\ k_i) \to \mathcal{DN}_c(Spec k)$$

and

$$\mathrm{colim}_i \mathcal{DN}(\mathrm{Spec}\ k_i) \to \mathcal{DN}(Spec k)$$

are equivalences in Cat_∞ *and* Pr^L *respectively.*

The same is true for $\eta = \lim_i U_i$ with $\eta \in X$ the generic point of an integral finite type k-scheme and the U_i are the nonempty open subsets of X:

$$\mathrm{colim}_i \mathcal{DN}_{(c)}(U_i) \to \mathcal{DN}_{(c)}(\eta).$$

Proof. As the functor Ind, from small idempotent complete stable ∞-categories to compactly generated presentable stable categories is an equivalence and that the inclusion functor $\mathrm{Pr}^L_\omega \to \mathrm{Pr}^L$ preserves colimits, the second statement follows from the first. By [IM22, Proposition 6.8] (or [IM22, Corollary 6.10] for the generic point case), the diagram $\mathcal{M} : I^{\triangleright} \to \mathrm{AbCAt}$ from I to the $(2,1)$-category of small abelian categories with exact functors sending $i \in I$ to $\mathcal{M}_{\mathrm{perv}}(\mathrm{Spec}\ k_i)$ and the cone point v to $\mathcal{M}_{\mathrm{perv}}(\mathrm{Spec}\ k)$ is a colimit diagram. By [BCKW19, Theorem 7.56] the functor sending an Abelian category to the ∞-category of bounded complexes up to homotopy is a left adjoint, hence commutes with colimits, so that we obtain a colimit diagram

$$\mathcal{K}^b \circ \mathcal{M} : I^{\triangleright} \to \mathrm{Cat}^{\mathrm{st}}_\infty.$$

Now as $\mathcal{D}^b(\mathcal{A})$ is a localisation of $\mathcal{K}^b(\mathcal{A})$ for any abelian category $\mathcal{A}$, and as the transitions of $\mathcal{K}^b \circ \mathcal{M}$ preserve quasi-isomorphisms, $\mathcal{D}^b \circ \mathcal{M}$ is a colimit diagram. □

Corollary 4.16. *Let $f : \mathrm{Spec}\ k \to \mathrm{Spec}\ \mathbb{Q}$ be a map of fields. Then the natural map $f^*\mathscr{N}_{\mathbb{Q}} \to \mathscr{N}_k$ is an equivalence. Consequently, for any scheme X of finite type over k if we denote by $p_X : X \to \mathrm{Spec}\ \mathbb{Q}$ the unique map (which has to factor through f), then $\mathscr{N}_X \simeq p_X^*\mathscr{N}_{\mathbb{Q}}$.*

Proof. The second claims follows from the first. Write $k = \mathrm{colim}_i k_i$ with $k_i/\mathbb{Q}$ of finite type. Then the same proof as [AGV22, Lemma 3.5.7], using continuity of both $\mathcal{DM}$ and $\mathcal{DN}$ for fields (Proposition 4.15) and that these categories a compactly generated gives that the natural map

$$\mathrm{colim}_i f_i^*\mathscr{N}_{\mathbb{Q}} \to \mathscr{N}_k$$

is an equivalence so that this reduces to the case where $k/\mathbb{Q}$ is of finite type. Then k is the generic point of an integral $\mathbb{Q}$-scheme, and again the same proof as [AGV22, Lemma 3.5.7], using continuity for the generic point of perverse Nori motives, reduces to show that $\pi_U^*\mathscr{N}_{\mathbb{Q}} \to \mathscr{N}_U$ is an equivalence for U of finite type over $\mathbb{Q}$, which is Lemma 4.14. □

Therefore now there is even less ambiguity for $\mathscr{N}_X = (X \to \mathrm{Spec}\ \mathbb{Q})^*\mathscr{N}_{\mathbb{Q}}$.

Proposition 4.17. *Let $X = \lim_i X_i$ be a limit of $\mathbb{Q}$-schemes with affine transitions. The natural morphism*

$$\mathrm{colim}_i \mathrm{Mod}_{\mathscr{N}}(\mathcal{DM}(X_i)) \to \mathrm{Mod}_{\mathscr{N}}(\mathcal{DM}(X)) \tag{4.17.1}$$

is an equivalence.

Proof. By [EHIK21, Lemma 5.1 (ii)], the functor

$$\mathrm{colim}_i \mathcal{DM}(X_i, \mathbb{Q}) \to \mathcal{DM}(X, \mathbb{Q}) \tag{4.17.2}$$

is an equivalence. Therefore, the assignment $i \mapsto (\mathcal{DM}(X_i, \mathbb{Q}), \mathscr{N}_{|X_i})$ is a colimit diagram in the category $\mathrm{Pr}^{\mathrm{Alg}}$ of [Lur22, Notation 4.8.5.10]. By [Lur22, Theorem 4.8.5.11], the functor $(\mathcal{C}, A)$ that sends a presentable symmetric monoidal category $\mathcal{C}$ together with an algebra object $A \in \mathcal{C}$ to the category $\mathrm{Mod}_A(\mathcal{C})$ has a right adjoint, hence preserves colimits. This is why the assignment $i \mapsto \mathrm{Mod}_{\mathscr{N}_{X_i}}(\mathcal{DM}(X_i, \mathbb{Q}))$ is a colimit diagram. We found a precise statement giving the symmetric monoidal structure in the paper by Ayoub, Gallauer and Vezzani. The precise statement is [AGV22, Lemma 3.5.6]. □

Proposition 4.18. *The functor*

$$\widetilde{\mathrm{Nor}^*} : \mathrm{Mod}_{\mathscr{N}}(\mathcal{DM}(X)) \to \mathcal{DN}(X) \tag{4.18.1}$$

is an equivalence for every finite type k-scheme X.

Proof. The functor is fully faithful by Lemma 4.14. The essential surjectivity is proven by Noetherian induction. Assume that $X = \mathrm{Spec}\ k$ is the spectrum of a field. As $\mathcal{DN}(\mathrm{Spec}\ k)$ is compactly generated it suffices to show that $\widetilde{\mathrm{Nor}^*}$ reaches all compact objects $M \in \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(k))$. As the functor is already fully faithful, by dévissage it suffices to check that any object of the heart $M \in \mathcal{M}_{\mathrm{perv}}(k)$ is in the image of $\widetilde{\mathrm{Nor}^*}$. Any such object has a weight filtration whose graded pieces are direct factors of $\mathrm{H}^{n+2j}(\pi_X)_*\mathbb{Q}_X(i)$ with $\pi_X : X \to \mathrm{Spec}\ K$ a smooth projective morphism by Arapura's [HMS17, Theorem 10.2.5]. Therefore by dévissage again it suffices to check that any such $\mathrm{H}^{n+2j}(\pi_X)_*\mathbb{Q}_X(i)$ is in the image. But as $(\pi_X)_*\mathbb{Q}_X(i)$ is pure of weight $-2i$, it is the direct sum of its $\mathrm{H}^n[n]$, thus is suffices to show that each $(\pi_X)_*\mathbb{Q}_X(i)$ is in the image, which is obvious by compatibility with the 6 operations.

Assume now that X is such that the statement is true for any proper closed subset of X. We take an object $M \in \mathcal{DN}(X)$. Let η be a generic point of X. By the case of a field, there exists $N \in \mathrm{Mod}_{\mathscr{N}}(\mathcal{DM}(\eta))$ such that $\widetilde{\mathrm{Nor}^*}(N) = M_\eta$. By Proposition 4.17 there exists an open subset U of X and an object $N' \in \mathrm{Mod}_{\mathscr{N}}(\mathcal{DM}(U))$ such that $(N')_\eta = N$. The functor $\mathrm{colim}_{\eta \in U}\mathcal{DN}(U) \to \mathcal{DN}(\eta)$ is fully faithful by proposition 4.15. Thus the isomorphism $\widetilde{\mathrm{Nor}^*}(N) \to M_\eta$ lifts to a smaller open subset V of U : we have $\widetilde{\mathrm{Nor}^*}((N')_{|V}) \simeq M_{|V}$. Denote by Z the reduced closed complement of V in X. By induction hypothesis, there exists a $N'' \in \mathrm{Mod}_{\mathscr{N}}(\mathcal{DM}(Z))$ such that $\widetilde{\mathrm{Nor}^*}(N'') = M_{|Z}$. The cofiber sequence $\widetilde{\mathrm{Nor}^*}(j_!(N')_{|V}) \to M \to \widetilde{\mathrm{Nor}^*}(i_*N'')$ with $j : U \to X$ and $i : Z \to X$ the immersion then ensures that M is the image of $\widetilde{\mathrm{Nor}^*}$, which finishes the proof. □

Remark 4.19. The same arguments would prove that the ∞-category

$$\mathcal{DH}^{\mathrm{geo}}(X) \subset \mathrm{Ind}\mathcal{D}^b(\mathrm{MHM}(X))$$

of geometric origin objects in the indization of the derived category of mixed Hodge modules is also the category of modules over some algebra $\mathscr{H}_X$ in $\mathcal{DM}(X)$. Note that the lack of continuity of Mixed Hodge modules would prevent things like $p_X^*\mathscr{H}_{\mathrm{Spec}\ \mathbb{Q}} \simeq \mathscr{H}_X$ of happening so one would have to be careful of which algebra is used and an additional argument over the generic point will be needed, similarly to the case of the Betti realisation. Similar ideas are developed in [Dre18] and the fact that one would obtain $(X \to \mathrm{Spec}\ k)^*\mathscr{H}_{\mathrm{Spec}\ k} \simeq \mathscr{H}_X$ identifies Drew's category as the full subcategory $\mathcal{DH}^{\mathrm{geo}}(X)$ of geometric origin mixed Hodge modules. His considerations with enriched categories of motives also have an application here: his results, together with the same proof as above, would show that his category **DH** of motivic enriched Hodge modules is the smallest full subcategory of $\mathrm{Ind}\mathcal{D}^b(\mathrm{MHM}(-))$ which is stable under truncation and filtered colimts, and that contains the f_*g^*H for $f : Y \to X$ proper, $g : Y \to \mathrm{Spec}\ \mathbb{C}$ the structural morphism and $H \in \mathcal{D}(\mathrm{IndMHS}^p)$.

Corollary 4.20. *The ∞-functor $X \mapsto \mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ has an unique extension $\mathcal{DN}_c(X)$ to quasi-compact and quasi-separated schemes of characteristic zero such that for all limit $X = \lim_i X_i$ of such schemes with affines transitions the natural functor*

$$\mathrm{colim}_i \mathcal{DN}_c(X_i) \to \mathcal{DN}_c(X)$$

is an equivalence of ∞-categories.

Proof. Let $\mathcal{C}$ be category of schemes that are of finite type over a field of characteristic zero. Let $\mathrm{Sch}_{\mathbb{Q}}$ be the category of quasi-compact and quasi-separated (qcqs) $\mathbb{Q}$-schemes, and $\iota : \mathcal{C} \to \mathrm{Sch}_{\mathbb{Q}}$ the inclusion. Nori motives, together with the pullback functoriality, define a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ which sends limits of schemes with affine transitions to colimits of ∞-categories by Proposition 4.17 together with the fact that the colimit is taken in Pr^L hence restricts to compact objects. We can denote by $\mathcal{DN}_c$ its left Kan extension along ι. □

Corollary 4.21. *Let X be a finite type k-scheme. The ∞-category $\mathcal{DN}(X)$ is compactly generated by the $f_\sharp\mathbb{Q}_Y(i)$ for $f : Y \to X$ smooth and $i \in \mathbb{Z}$.*

Proof. Indeed, this the case for $\mathcal{DM}(X)$, thus for $\mathrm{Mod}_{\mathscr{N}_X}(\mathcal{DM}(X)) \simeq \mathcal{DN}(X)$ by [Iwa18, Lemma 2.6]. □

We finish with the proof of the following natural corollary, whose proof was surprisingly harder than expected for a result that the author took for granted for a long time.

Corollary 4.22. *Let X be a quasi-projective k-scheme. The two functors*

$$\mathcal{DM}_c(X) \to \mathcal{M}_{\mathrm{perv}}(X)$$

given by ${}^{\mathrm{p}}\mathrm{H}^0 \circ \mathrm{Nor}^*$ *and the universal functor* $\mathrm{H}_{\mathrm{univ}}$ *are canonically equivalent.*

Proof. First, note that the functor ${}^{\mathrm{p}}\mathrm{H}^0 \circ \mathrm{Nor}^*$ gives a factorisation of the ${}^{\mathrm{p}}\mathrm{H}^0$ of the Betti realisation of étale motives

$$\mathcal{DM}_c(X) \to \mathrm{Perv}(X).$$

By the universal property of perverse Nori motives, this induces a canonical functor

$$F : \mathcal{M}_{\mathrm{perv}}(X) \to \mathcal{M}_{\mathrm{perv}}(X)$$

such that $R_B \circ F \simeq R_B$ and ${}^{\mathrm{p}}\mathrm{H}^0 \circ \mathrm{Nor}^* \simeq F \circ \mathrm{H}_{\mathrm{univ}}$. Now the compatibility of Nor^* with the operations ensures that for any perverse t-exact operation h the functor F commutes with the operation h. Hence F commutes with i_* for i a closed immersion and for $f^*[d]$ for f a smooth map of relative dimension d. We will still denote by F the functor induced by it on $\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$. As F commutes with i_*, there exists an exchange morphism $i^* \circ F \to F \circ i_*$. Applying the Betti (or ℓ-adic in the case the field k is too big) realisation where this exchange morphism is an equivalence, we see that F commutes with i^*. The same argument works to show that F commutes with f_* for smooth f. For a smooth morphism f, there is a natural transformation $f_\sharp \circ F \to F \circ f_\sharp$ which is an equivalence after realisation, hence F also commutes with the $f_\sharp$ functors. It also commutes with external tensor product because is it perverse t-exact, hence with the internal tensor product which is obtained from the external one by pulling back along the diagonal.

The functor F induces a compact and colimit preserving functor

$$F : \mathrm{Ind}\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X)) \to \mathrm{Ind}\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$$

which commutes with all pullbacks, pushforwards and left adjoint to pushforward by smooth maps. Denote by G its right adjoint. We claim that the natural transformation

$$\mathrm{Id} \to GF$$

is an equivalence. Indeed, by Corollary 4.21 we have that $\mathrm{Ind}\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$ is generated by colimits by the $f_\sharp\mathbb{Q}_Y(i)$ for $f : Y \to X$ a smooth map and $i \in \mathbb{Z}$. This means that it suffices to check that for $M \in \mathrm{Ind}\mathcal{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$, the map

$$\mathrm{Map}(f_\sharp\mathbb{Q}_Y(i), M) \to \mathrm{Map}(f_\sharp\mathbb{Q}_Y(i), GF(M))$$

is an equivalence. As G and F commute with filtered colimits and $f_\sharp \mathbb{Q}_Y(i)$ is compact, we can assume that M is as well compact. Moreover, compact objects are constructed by extensions, finite limits, finite colimits and direct factors from the same $f_\sharp \mathbb{Q}_Y(i)$, thus by dévissage we can assume that $M = g_\sharp \mathbb{Q}_Z(j)$ for some smooth map $g : Z \to X$ and $j \in \mathbb{Z}$. Using that G is a left adjoint and that F commutes with the operations, the above map is equivalent to the map

$$\mathrm{Map}(f_\sharp \mathbb{Q}_Y(i), g_\sharp \mathbb{Q}_Z(j)) \to \mathrm{Map}(f_\sharp F(\mathbb{Q}_Y)(i), g_\sharp F(\mathbb{Q}_Z)(j))$$

and as $F(\mathbb{Q}) = \mathbb{Q}$, this finishes the proof.

Now that F is fully faithful, we note that it is an equivalence whenever $X = \mathrm{Spec}\, k$ is the spectrum of a field by [HMS17, Corrolary 10.1.7]. Thus an easy dévissage using the localisation sequence and continuity proves that F is essentially surjective, finishing the proof that F is an isomorphism. Now, as F commutes with the operations, we have by unversality of $\mathcal{DM}$ that $F \circ {}^{\mathrm{p}}\mathrm{H}^0 \circ \mathrm{Nor}^* \simeq {}^{\mathrm{p}}\mathrm{H}^0 \circ \mathrm{Nor}^*$, thus $F \circ F \circ \mathrm{H}_{\mathrm{univ}} \simeq {}^{\mathrm{p}}\mathrm{H}^0 \circ \mathrm{Nor}^*$ and the universal property of $\mathcal{M}_{\mathrm{perv}}(X)$ tells us that $F \circ F \simeq F$, so that $F \simeq \mathrm{Id}$. □

A Coefficient systems.

Recall the following definition of Drew and Gallauer in [DG22] :

Definition A.1. A functor $C : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})$ taking values in symmetric monoidal stable ∞-categories and exact symmetric monoidal functors is called a *coefficient system* if it satisfies the following properties.

1. **(Pushforwards)** For every $f : Y \to X$ in Sch_k, the pullback functor f^* admits a right adjoint $f_* : C(Y) \to C(X)$.
2. **(Internal homs)** For every $X \in \mathrm{Sch}_k$, the symmetric monoidal structure on $C(X)$ is closed.
3. For each smooth morphism $p : Y \to X \in \mathrm{Sch}_k$, the functor $p^* : C(X) \to C(Y)$ admits a left adjoint $p_\sharp$, and:
   (a) **(Smooth base change)** For each cartesian square

   $$\begin{array}{ccc} Y' & \xrightarrow{p'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{p} & X \end{array}$$

   in Sch_k, the exchange transformation $p'_\sharp (f')^* \to f^* p_\sharp$ is an equivalence.
   (b) **(Smooth projection formula)** The exchange transformation

   $$p_\sharp(p^*(-) \otimes -) \to - \otimes p_\sharp(-)$$

   is an equivalence of functors $C(X) \times C(Y) \to C(Y)$.

4. **(Localization)** For each closed immersion $Z \to X$ in Sch_k with complementary open immersion $j : U \to X$, the square

$$\begin{array}{ccc} C(Z) & \xrightarrow{i_*} & C(X) \\ \downarrow & & \downarrow{\scriptstyle j^*} \\ 0 & \longrightarrow & C(U) \end{array}$$

is cartesian in $\mathrm{Cat}_\infty^{\mathrm{st}}$.

5. For each $X \in \mathrm{Sch}_k$, if $\pi_{\mathbb{A}^1} : \mathbb{A}^1_X \to X$ denotes the canonical projection with zero section $s : X \to \mathbb{A}^1_X$, then:

6. **($\mathbb{A}^1$-homotopy invariance)** The functor $\pi^*_{\mathbb{A}^1} : C(X) \to C(\mathbb{A}^1_X)$ is fully faithful.

7. **(T-stability)** The composite $\pi_{\mathbb{A}^1,\sharp} s_* : C(X) \to C(X)$ is an equivalence.

A morphism of coefficient systems is a natural transformation $\phi : C \to C'$ such that for each smooth morphism $p : Y \to X$ in Sch_k, the exchange transformation

$$p_\sharp \phi_Y \to \phi_X p_\sharp$$

is an equivalence. This defines a sub ∞-category $\mathrm{CoSy}_k \subset \mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}}))$. Denote by $\mathrm{CoSy}_k^{\mathrm{c}}$ the category of coefficients that take values in cocomplete categories and such that all pullbacks are colimit preserving.

Recall the main theorem of [DG22]:

Theorem A.2 (Drew-Gallauer)**.** *The stable $\mathbb{A}^1$-homotopy ∞-category defines an object $\mathcal{SH} \in \mathrm{CoSy}_k^{\mathrm{c}}$. It is the initial object.*

Definition A.3. We will denote by $\mathrm{CoSy}_k^{\mathrm{Zar}}$ (*resp.* by $\mathrm{CoSy}^{\text{ét}}$) the full subcategory of coefficients that consist of objects such that the underlying functor

$$\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

is a Zariski (*resp.* an étale) sheaf.

Definition A.4. We let QPCoSy_k and $\mathrm{QPCoSy}_k^{\mathrm{Zar}}$ the ∞-categories defined in the same terms as Definition A.1 but with functors

$$\mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})$$

with source quasi-projective varieties instead of finite type k-schemes.

Proposition A.5. *The restriction functor*

$$\mathrm{CoSy}_k^{\mathrm{Zar}} \to \mathrm{QPCoSy}_k^{\mathrm{Zar}}$$

is an equivalence of ∞-categories with quasi-inverse the right Kan extension. The same is true for any topology finer than the Zariski topology.

Proof. It suffices to deal with the Zariski topology. Restriction and right Kan extension give an equivalence of categories between Zariski sheaves

$$\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Var}_k, \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})) \simeq \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sch}_k, \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}}))$$

because any finite type k-scheme admits a Zariski covering by affine schemes. Therefore it suffices to check that if $C \in \mathrm{QPCoSy}_k^{\mathrm{Zar}}$, its right Kan extension to $\mathrm{Sch}^{\mathrm{op}}$ is an object of $\mathrm{CoSy}_k^{\mathrm{Zar}}$, and that a morphism in $\mathrm{QPCoSy}_k^{\mathrm{Zar}}$ gives a morphism in $\mathrm{CoSy}_k^{\mathrm{Zar}}$ after Kan extension.

We denote by $\mathcal{D}$ the right Kan extension of the functor

$$C : \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

along the inclusion $\mathrm{Var}_k^{\mathrm{op}} \to \mathrm{Sch}_k^{\mathrm{op}}$. Because C is an étale sheaf, for $\in \mathrm{Sch}_k$ we may choose a Zariski covering $p : U \to X$ with U affine and as p is quasi-projective, all $U_n := U^{\times_X n+1}$ are quasi-projective and we have

$$\mathcal{D}(X) \xrightarrow{\sim} \lim_{\Delta} C(U_n)$$

in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$. But then if $f : Y \to X$ is a smooth morphism, base change induces a map of simplicial schemes $f_\bullet : V_\bullet \to U_\bullet$ with $V_n = U_n \times_X Y$ and we also have

$$\mathcal{D}(Y) \xrightarrow{\sim} \lim_{\Delta} C(V_n)$$

in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$. Moreover if $p_{U,n}$ (*resp.* $p_{V,n}$) is the map $U_n \to X$ (*resp.* $V_n \to Y$) the smooth base change theorem in C ensures that the family of functors

$$(f_n)_\sharp p_{V,n}^* \colon \mathcal{D}(Y) \to C(U_n) \tag{A.5.1}$$

defines a functor

$$f_\sharp \colon \mathcal{D}(Y) \to \mathcal{D}(X).$$

As limits are computed level wise in Cat_∞, it is easy to see that $f_\sharp$ is a left adjoint of f^* and if one consider those limits in $\mathrm{Cat}_{\mathcal{D}(X)}^{\mathrm{perf}}$ the ∞-category of modules over $\mathcal{D}(X)$ in the ∞-category of idempotent complete small stable ∞-categories, we obtain that the projection formula holds for $f_\sharp$ (the projection formula for each $(f_n)_\sharp$ implies that the functors in (A.5.1) are $\mathcal{D}(X)$-linear). Moreover, for each cartesian square in Sch_k of the form

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ S' & \xrightarrow{f} & S \end{array}$$

with f a smooth map, by taking a Zariski open covering $\pi_U : U \to S$ with U affine and doing a base change we obtain a pullback square in Var_k of the form

$$\begin{array}{ccc} V' & \xrightarrow{b'} & V \\ {\scriptstyle a'}\downarrow & \lrcorner & \downarrow{\scriptstyle a} \\ U' & \xrightarrow{b} & U \end{array} .$$

We will denote by $\pi_V : V \to X$ the base change of π_U, and similarly for V' and U'. The fact that the exchange transformation

$$f'_\sharp (g')^* \to g^* f_\sharp$$

is an isomorphism can be checked after applying the functor π_X^* which is conservative. But now we have:

$$\begin{aligned} \pi_X^* f'_\sharp (g')^* &\overset{\text{def}}{\simeq} b'_\sharp \pi_{X'}^* (g')^* \\ &\simeq b'_\sharp (a')^* \pi_{U'}^* \\ &\overset{\text{SmBC}}{\simeq} a^* b_\sharp \pi_{U'}^* \\ &\overset{\text{def}}{\simeq} a^* \pi_U^* f_\sharp \\ &\simeq \pi_X^* g^* f_\sharp. \end{aligned}$$

The existence of pushforwards and internal $\mathscr{H}om$ is proven in a similar way to the existence of the $f_\sharp$. The only thing to check in order to obtain those functors as functors to a limit of ∞-categories is that over quasi-projective schemes these functors commute with pullbacks by Zariski covering so that they define a compatible system as in (A.5.1). This is the case because these pullbacks π^* are right adjoints to $\pi_\sharp$ and the smooth base change and the projection formula ensures the commutativity of $\pi_\sharp$ with the left adjoints of internal $\mathscr{H}om$ and pushforwards. Localisation follows from the fact that the category $\mathrm{Sq}^{\mathrm{cart}}(\mathrm{Cat}_\infty)$ of cartesian squares in Cat_∞ is closed under limits in the category of commutative squares of ∞-categories. The properties of $\mathbb{A}^1$-invariance and T-stability can be checked Zariski locally thanks to the smooth projection formula and the smooth base change, hence they hold. The same is true for extensions of morphisms of coefficient systems. □

Finally recall the following theorem:

Theorem A.6 (Ayoub-Cisinski-Déglise-Röndings-Voeveodsky)**.** *Any coefficient system affords a six functors formalism.*

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