Mixed Hodge modules on stacks

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Abstract

Using the ∞-categorical enhancement of mixed Hodge modules constructed by the author in a previous paper, we explain that mixed Hodge modules canonically extend to algebraic stacks, together with all the 6 operations and weights. We also prove that Drew's approach to motivic Hodge modules gives an ∞-category that embeds fully faithfully in mixed Hodge modules, and we identify the image as mixed Hodge modules of geometric origin.

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Introduction.

Let X be a complex algebraic variety and $n \in \mathbb{N}$ be an integer. Deligne's work in [Del74] gives a polarisable mixed Hodge structure on the singular cohomology $\mathrm{H}^n_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$ of the complex points of X, seen as an analytic variety. M. Saito's category of mixed Hodge modules ([Sai90b]) on X is an Abelian category $\mathrm{MHM}(X)$ modeled on perverse sheaves which is relative version of polarisable mixed Hodge structures. It's derived category $\mathrm{D}^b(\mathrm{MHM}(X))$ is endowed with the 6 sheaf operations an any complex of mixed Hodge modules K has an underlying complex $\mathrm{rat}(K)$ of perverse sheaves. The mixed Hodge structure on $\mathrm{H}^n_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$ can be recovered as $\mathrm{H}^n(f_*\mathbb{Q}_X)$ where

$$f_*\colon \mathrm{D}^b(\mathrm{MHM}(X)) \to \mathrm{D}^b(\mathrm{MHM}(\mathrm{Spec}(\mathbb{C}))) \simeq \mathrm{D}^b(\mathrm{MHM}^p_{\mathbb{C}})$$

is the pushforward of the map $f : X \to \mathrm{Spec}(\mathbb{C})$ and $\mathbb{Q}_X \in \mathrm{D}^b(\mathrm{MHM}(X))$ is the unit for the tensor product, whose underlying sheaf of perverse sheaves is the constant sheaf of $\mathbb{Q}$-vector spaces $\mathbb{Q}_X \in \mathrm{Sh}(X(\mathbb{C}), \mathbb{Q})$ under Beilinson's ([Bei87b]) isomorphism $\mathrm{D}^b(\mathrm{Perv}(X, \mathbb{Q})) \simeq \mathrm{D}^b_c(X(\mathbb{C})^{\mathrm{an}}, \mathbb{Q})$. This point of view is very powerful as the formalism of the 6 operations is useful to make computations and reductions.

If $\mathfrak{X}$ is a reasonable algebraic stack (say a global quotient stack or a stack exhausted étale locally by such), one can construct a mixed Hodge structure on $\mathrm{H}^n_{\mathrm{sing}}(\mathfrak{X}(\mathbb{C}), \mathbb{Q})$ either by hyperdescent as in [Del74, Section 6.1] or by using exhaustions as in [Dav24, Section 5.2]. One purpose of this note (see Section 3.1) is to give an extension of M. Saito's derived category, together with the 6-operations, to algebraic stacks so that the above mixed Hodge structures can be recovered as $\mathrm{H}^n(f_*\mathbb{Q}_{\mathfrak{X}})$ with $f : \mathfrak{X} \to \mathrm{Spec}(\mathbb{C})$ the structural map:

Theorem (Theorem 3.1,Corollary 3.21 and Theorem 3.32)**.** *There exists a canonical extension of the derived category of mixed Hodge modules to algebraic stacks over the complex numbers. It has the* 6*-operations, nearby cycles, and a notion of weights. Over stacks with affine stabilizers this notion of weights give rise to a weight structure* à la *Bondarko.*

The proof relies on the ∞-categorical enhancement of mixed Hodge modules obtained in [Tub23], and then on Liu and Zheng's work ([LZ17]) on extension of formalisms of 6 operations to stacks. The construction of nearby cycles is based on a motivic construction of the unipotent nearby cycles functor considered in [CHS24]. The existence of the weight structure *à la* Bondarko ([Bon11]) results of a computation of the weights of pushforwards under morphisms from stacks with affine stabilizers by Sun ([Sun12]) in the context of ℓ-adic sheaves on stacks over finite fields, whose proof is easily adapted to the Hodge setting.

There is another approach to giving a relative version of mixed Hodge structures. In [Dre18], Drew constructs an ∞-category of *motivic Hodge modules* $\mathrm{DH}(X)$ which is endowed with the 6 operations as well as with a notion of weights. If $X = \mathrm{Spec}\,\mathbb{C}$ this category embeds fully faithfully in $\mathrm{D}(\mathrm{IndMHM}^p_{\mathbb{C}})$ the derived category of the indization of mixed Hodge modules. This construction has the advantage of being quite straightforward: one consider the commutative algebra $\mathcal{H}$ in the ∞-category of Voevodsky étale motives that represents Hodge cohomology, and then one takes modules over this algebra. In comparison, M. Saito's construction is very delicate and requires a lot of attention in order to work. The major drawback of Drew's construction is that there was no easy construction of a t-structure, hence one looses access to an abelian category. The second purpose of this note is to prove that Drew's category gives the right thing: it is endowed with a t-structure and it embeds fully faithfully in the derived category of mixed Hodge modules. We also identify its image.

Theorem (Theorem 2.7)**.** *Let X be a finite type scheme over the complex numbers. The ∞-category of motivic Hodge modules on X of Drew embeds in the derived category of ind-mixed Hodge modules on X. Its image is the category generated under shifts, and colimits by objects of the for $f_*\mathbb{Q}_Y$ with $f : Y \to X$ a proper morphism.*

There is also an improved version of this theorem with enriched motivic Hodge modules, that reach the ∞-category generated under shifts and colimits by objects of the form f_*g^*H with $f : Y \to X$ proper, $g : Y \to \mathrm{Spec}\,\mathbb{C}$ the structural morphism and $H \in \mathrm{MHS}^p_{\mathbb{C}}$ a polarisable rational mixed Hodge structure over $\mathbb{C}$. We refer the reader to Section 2 for more details. To prove this result, we use that by the work of Drew and Gallauer ([DG22]) the ∞-categorical enhancement of mixed Hodge modules provides a realisation functor

$$\rho_H : \ \mathrm{DM}_{\text{ét}} \to \mathrm{Ind}\mathrm{D}^b(\mathrm{MHM}(-))$$

from the presentable ∞-category of rational étale Voevodsky motives to the indization of the derived category of mixed Hodge modules, that commutes with the operations and is colimit preserving. By abstract nonsense this functor will factors through Drew's category of mixed Hodge modules, and gives a fully faithful functor $\mathrm{DH} \to \mathrm{Ind}\mathrm{D}^b(\mathrm{MHM})$. The identification of the image is inspired by the proof of Ayoub in the Betti case ([Ayo22, Theorem 1.98]), and relies on the semi-simplicity of smooth and proper pushforwards of pure mixed Hodge modules.

Organization of the paper.

In the first Section 1 of this note we recall how to construct the ∞-categorical enhancement of the 6 operations for mixed Hodge modules, and why this gives a Hodge realisation of étale motives. In the second Section 2 we show that Drew's construction embeds fully faithfully in mixed Hodge modules. In the last Section 3.1 we explain how to extend mixed Hodge modules on stacks, finishing with a comparison to existing constructions.

1 Recollections on ∞-categorical enhancements of mixed Hodge modules.

1.1 Construction of the enhancement.

In [Tub23], we proved that Saito's construction of the triangulated bounded derived category of algebraic mixed Hodge modules, together with the 6 operations, can be enhanced to the world of ∞-categories. Let us recall how this works:

For every separated finite type $\mathbb{C}$-scheme X, the bounded derived category $\mathrm{D}^b(\mathrm{MHM}(X))$ of mixed Hodge modules carries a natural *ordinary* t-structure (called the *constructible* t-structure in *loc. cit.* but we prefer the name ordinary because all complexes on $\mathrm{D}^b(\mathrm{MHM}(X))$ are constructible) which is characterized by the fact that

$$\mathrm{D}^b(\mathrm{MHM}(X))^{t_{\mathrm{ord}}\in[a,b]} = \{K \in \mathrm{D}^b(\mathrm{MHM}(X)) \mid \forall x \in X,\ x^*K \in \mathrm{D}^{[a,b]}(\mathrm{MHS}^p_{\mathbb{C}})\}$$

for every $-\infty \leqslant a \leqslant b \leqslant +\infty$. If we endow $\mathrm{D}^b_c(X(\mathbb{C})^{\mathrm{an}}, \mathbb{Q})$ with its canonical t-structure induced by the inclusion

$$\mathrm{D}^b_c(X(\mathbb{C})^{\mathrm{an}}, \mathbb{Q}) \subset \mathrm{D}(\mathrm{Sh}(X(\mathbb{C})^{\mathrm{an}}, \mathbb{Q}))$$

then the "underlying $\mathbb{Q}$-structure functor"

$$\mathrm{rat}\colon \mathrm{D}^b(\mathrm{MHM}(X)) \to \mathrm{D}^b_c(X(\mathbb{C})^{\mathrm{an}}, \mathbb{Q})$$

is t-exact if the left hand side is endowed with the ordinary t-structure. Denote by $\mathrm{MHM}_{\mathrm{ord}}(X)$ the heart of the ordinary t-structure. The crucial result in [Tub23] is the following, whose proof is adapted from Nori's proof of the analogous result for $\mathrm{D}^b_c(X(\mathbb{C})^{\mathrm{an}}, \mathbb{Q})$:

Theorem 1.1 ([Tub23, Corollary 2.19]). *The canonical functor*

$$\mathrm{D}^b(\mathrm{MHM}_{\mathrm{ord}}(X)) \to \mathrm{D}^b(\mathrm{MHM}(X))$$

is an equivalence.

Using this, because pullbacks f^* are t-exact for the ordinary t-structure, it is not hard to see that they are derived functors, hence that they canonically have a ∞-categorical enhancement, and the same can be said for the tensor product. The enhancements of the other operations arises by adjunction: if a ∞-functor between stable ∞-categories has an adjoint on the homotopy triangulated category, it has an ∞-categorical adjoint by [NRS20, Theorem 3.3.1].

In particular, we have a functor

$$\mathrm{D}^b(\mathrm{MHM}(-)) : \mathrm{Sch}^{\mathrm{op}}_{\mathbb{C}} \to \mathrm{CAlg}(\mathrm{St})$$

taking values in the ∞-category of stably symmetric monoidal ∞-categories and symmetric monoidal exact functors. It sends a map $f : Y \to X$ of finite type $\mathbb{C}$-schemes (say, separated) to an ∞-functor f^* lifting Saito's pullback functor on the homotopy category. For smooth f, the functor f^* admits a left adjoint $f_\sharp$, and that moreover $\mathrm{D}^b(\mathrm{MHM}(-))$ satisfies the usual axioms considered in motivic context: $\mathbb{A}^1$-invariance, $\mathbb{P}^1$-stability, smooth base change, proper base change *etc*. In particular, the underlying homotopy functor is a *motivic triangulated category* in the sense of [CD19, Definition 2.4.45], it satisfies moreover h-hyperdescent because it is $\mathbb{Q}$-linear and satisfies étale descent, localisation and the proper base change (see [CD19, Theorem 3.3.37]). We now consider the functor

$$\mathrm{D}_H(-) := \mathrm{Ind}\mathrm{D}^b(\mathrm{MHM}(-)) : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{St}})$$

taking values in *presentable* ∞-categories.

Remark 1.2. The above functor extends tautologically to diagrams of schemes, hence we can evaluated D_H on a simplicial scheme to obtain a cosimplicial diagram of ∞-categories. Using the fact that it is easy (see [Sai90b, Remark before 4.6]) to compare the mixed Hodge structures on $\mathrm{H}^n_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$ constructed by Deligne and Saito (under the equivalence $\mathrm{MHM}(\mathrm{Spec}(\mathbb{C})) \simeq \mathrm{MHS}^p_{\mathbb{C}}$) when X is a closed subset of a smooth variety, h-hyperdescent of the derived category of mixed Hodge modules gives a simple proof that in fact the two mixed Hodge structures are the same for a general complex variety X. This result was known, but the proof is quite involved (see [Sai00]).

Remark 1.3. Everything we do in this note would probably hold more generally for arithmetic mixed Hodge modules over varieties defined over a subfield k of $\mathbb{C}$. They are considered in [Sai06, Examples 1.8 (ii)], and our work in [Tub23], thus the proofs of this note, would probably work *verbatim* in this slightly more general context.

1.2 Hodge realisation of étale motives.

The main motivation for the ∞-categorical lifting of mixed Hodge modules was that this was the only obstruction for the existence of a realisation functor from Voevodsky motives that commutes with all the operations. We will deal here with the étale version, with rational coefficients. Recall that the ∞-category of étale motives with rational coefficients over a scheme X is defined as a formula by

$$\mathrm{DM}_{\text{ét}}(X) := \mathrm{Sh}^{\wedge}_{\text{ét},\mathbb{A}^1}(\mathrm{Sm}_X, \mathrm{Mod}_{\mathbb{Q}})[\mathbb{Q}(1)^{\otimes -1}].$$

This means that to construct $\mathrm{DM}_{\text{ét}}(X)$, one consider étale hypersheaves on smooth X-schemes with values in $\mathrm{Mod}_{\mathbb{Q}} \simeq \mathrm{D}(\mathbb{Q})$ that are $\mathbb{A}^1$-invariant, and then one invert the object $\mathbb{Q}(1) = (M(\mathbb{P}^1_X)/M(\infty_X))[-2]$ for the tensor product, with M the Yoneda embedding. The formula clearly gives an universal property of the presentable symmetric monoidal ∞-category $\mathrm{DM}_{\text{ét}}(X)$. By the work of Drew and Gallauer [DG22] in fact this universality of $\mathrm{DM}_{\text{ét}}(X)$ induces a universal property of the functor $\mathrm{DM}_{\text{ét}}$ on schemes of finite type over some basis S. Together with [CD19, Theorem 4.4.25], taking $S = \mathrm{Spec}(\mathbb{C})$ one obtains:

Theorem 1.4 ([Tub23, Theorem 4.4])**.** *There exists a Hodge realisation*

$$\rho_H \colon \mathrm{DM}_{\text{ét}} \to \mathrm{D}_\mathrm{H}$$

on finite type $\mathbb{C}$-schemes that commutes with the 6-operations. Moreover the composition with the functor

$$\mathrm{rat} \colon \mathrm{D}_H \to \mathrm{Ind}\mathrm{D}^b_c(-, \mathbb{Q})$$

gives the Betti realisation.

We will use this functor in the next section to compare mixed Hodge modules with Drew's approach, and at the end of this note to obtain the computation of the cohomology of a quotient stack with exhaustions in a easy way.

2 Mixed Hodge modules of geometric origin.

In this section, we prove that Drew's approach in [Dre18] to motivic Hodge modules gives a full subcategory of the derived category of mixed Hodge modules, stable under truncation, and thus has a t-structure.

2.1 Motivic Hodge modules.

As recalled above, we have a Hodge realisation

$$\rho_H\colon\ \mathrm{DM}_{\text{ét}} \to \mathrm{D}_H$$

compatible with all the operations. This functor is $\mathrm{Mod}_{\mathbb{Q}}$-linear. The target D_H is naturally valued in $\mathrm{D}_H(\mathbb{C}) = \mathrm{Ind}\mathrm{D}^b(\mathrm{MHS}^p_{\mathbb{C}})$-linear presentable ∞-categories, where $\mathrm{MHS}^p_{\mathbb{C}}$ is the abelian category of polarisable mixed Hodge structures over $\mathbb{C}$ (with rational coefficients). Thus the above realisation has a natural enrichment

$$\rho_{\mathbf{H}}\colon \mathbf{DM} \to \mathrm{D}_H$$

where $\mathbf{DM} := \mathrm{DM}_{\text{ét}} \otimes_{\mathrm{Mod}_{\mathbb{Q}}} \mathrm{D}_H(\mathbb{C})$ is the $\mathrm{D}_H(\mathbb{C})$-linearisation of $\mathrm{DM}_{\text{ét}}$. It can be computed as

$$\mathbf{DM}(X) = \mathrm{DM}_{\text{ét}}(X) \otimes_{\mathrm{Mod}_{\mathbb{Q}}} \mathrm{D}_H(\mathbb{C}),$$

but can also be put inside the definition:

$$\mathbf{DM}(X) \simeq \mathrm{Shv}^{\wedge}_{\mathbb{A}^1,\text{ét}}(\mathrm{Sm}_X, \mathrm{D}_H(\mathbb{C}))[\mathbb{Q}(1)^{\otimes -1}].$$

Thus the ∞-category $\mathbf{DM}$ is the $\mathbb{P}^1$-stabilisation of the $\mathbb{A}^1$-invariant étale hypersheaves on Sm_X with values in $\mathrm{D}_H(\mathbb{C})$. In particular, it also affords the 6 operations and the canonical functor $\mathrm{DM}_{\text{ét}} \to \mathbf{DM}$ commutes with them. For each finite type $\mathbb{C}$-scheme X, the functors

$$\rho_{H,X}\colon\ \mathrm{DM}_{\text{ét}}(X) \to \mathrm{D}_H(X)$$

and

$$\rho_{\mathbf{H},X}\colon \mathbf{DM}(X) \to \mathrm{D}_H(X)$$

are colimit preserving symmetric monoidal functors, hence they have lax symmetric monoidal right adjoints $\rho_*^{H,X}$ and $\rho_*^{\mathbf{H},X}$ that create commutative algebras

$$\mathcal{H}_X := \rho_*^{H,X}\mathbb{Q}_X \in \mathrm{CAlg}(\mathrm{DM}_{\text{ét}}(X))$$

and

$$\boldsymbol{\mathcal{H}}_X := \rho_*^{\mathbf{H},X}\mathbb{Q}_X \in \mathrm{CAlg}(\mathbf{DM}(X))$$

in étale motives. Because the right adjoints commute with pushforwards, the counit maps induces algebra maps

$$\mathcal{H}_{X/\mathbb{C}} := \pi_X^*\mathcal{H}_{\mathbb{C}} \to \mathcal{H}_X$$

and

$$\boldsymbol{\mathcal{H}}_{X/\mathbb{C}} := \pi_X^*\boldsymbol{\mathcal{H}}_{\mathbb{C}} \to \boldsymbol{\mathcal{H}}_X,$$

where $\pi_X : X \to \operatorname{Spec}\mathbb{C}$ is the structural map. We will show below that those maps are in fact equivalences.

Definition 2.1 (Drew)**.** The ∞-category of motivic Hodge modules over X is

$$\mathrm{DH}(X) := \mathrm{Mod}_{\mathcal{H}_{X/\mathbb{C}}}(\mathrm{DM}_{\text{ét}}(X)).$$

There is an enriched version, that we will call the ∞-category of enriched mixed Hodge modules over X, defined as

$$\mathbf{DH}(X) := \mathrm{Mod}_{\boldsymbol{\mathcal{H}}_{X/\mathbb{C}}}(\mathbf{DM}(X)).$$

Notation 2.2. *As the reader begins to guess, both situations, motivic Hodge modules and enriched motivic Hodge modules are parallel. Thus from now on except for the important results we will only deal with motivic Hodge modules, the proofs in the enriched case being the same. This is only to avoid doubling the size of this note, and the extensive use of bold.*

The functor $\mathrm{DM}_{\text{ét}}$ is in fact valued in the ∞-category of $\mathrm{DM}_{\text{ét}}(\mathbb{C})$-linear presentable ∞-categories, and the construction $\mathrm{DH} = \mathrm{Mod}_{\mathcal{H}_{-/\mathbb{C}}}(\mathrm{DM}_{\text{ét}})$ can be rewritten

$$\mathrm{DH} = \mathrm{DM}_{\text{ét}} \otimes_{\mathrm{DM}_{\text{ét}}(\mathbb{C})} \mathrm{DH}(\mathbb{C}).$$

As $\mathrm{DM}_{\text{ét}}(\mathbb{C})$ and $\mathrm{DH}(\mathbb{C})$ are rigid, the ∞-functor

$$- \otimes_{\mathrm{DM}_{\text{ét}}(\mathbb{C})} \mathrm{DH}(\mathbb{C}) \colon \mathrm{Pr}^L_{\mathrm{DM}_{\text{ét}}(\mathbb{C})} \to \mathrm{Pr}^L_{\mathrm{DH}(\mathbb{C})} \tag{2.2.1}$$

sending a $\mathrm{DM}_{\text{ét}}(\mathbb{C})$-linear presentable ∞-category $\mathcal{C}$ to $\mathcal{C} \otimes_{\mathrm{DM}_{\text{ét}}(\mathbb{C})} \mathrm{DH}(\mathbb{C}) \simeq \mathrm{Mod}_{\mathcal{H}_{\mathbb{C}}}(\mathcal{C})$ has an (∞-2)-categorical enhancement thanks to [HSS17, Section 4.4]. In particular any adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \quad (\perp)$$

between $\mathrm{DM}_{\text{ét}}(\mathbb{C})$-linear presentable ∞-categories, such that the right adjoint G itself commutes with colimits and is $\mathrm{DM}_{\text{ét}}(\mathbb{C})$-linear (this is automatic if F preserves compact objects by [HSS17, Proposition 4.9] because $\mathrm{DM}_{\text{ét}}(\mathbb{C})$ and $\mathrm{DH}(\mathbb{C})$ are rigid), the image under (2.2.1) is again an internal $\mathrm{DH}(\mathbb{C})$-adjunction, which means that $G \otimes \mathrm{DH}(\mathbb{C})$ is the right adjoint to $F \otimes \mathrm{DH}(\mathbb{C})$. Now it turns out that all properties of coefficients systems ([DG22]) or motivic categories ([CD19]) are properties of internal adjunctions. For example the projection formula for smooth morphism exactly expresses that $f_\sharp$ and f^* are $\mathrm{DM}_{\text{ét}}(S)$ where S is the basis. The property of $\mathbb{A}^1$-invariance of such system of ∞-categories also is expressed as the isomorphism $p_\sharp p^* \to \mathrm{Id}$ is an equivalence, where $p : \mathbb{A}^1_S \to S$ is the projection, and so on. Thus our functor

$$\mathrm{DH} \colon \mathrm{Sch}^{\mathrm{op}}_{\mathbb{C}} \to \mathrm{Pr}^L_{\mathrm{St}}$$

is naturally a coefficient system in the sense of Drew and Gallauer, and therefore affords the 6 operations in a way that is compatible to the functor $\mathrm{DM}_{\text{ét}} \to \mathrm{DH}$, and has h-descent. A more hands on proof of this result had been given by Drew in [Dre18, Theorem 8.10].

2.2 Embedding in mixed Hodge modules.

The Hodge realisation naturally factors as

$$\mathrm{DM}_{\text{ét}} \xrightarrow{\otimes \mathcal{H}_{\mathbb{C}}} \mathrm{DH} \xrightarrow{\rho_H} \mathrm{D}_H$$

where all functors commute with the operations and all categories are compactly generated on schemes. Indeed one can see it in the following way:

$$\mathrm{DM}_{\text{ét}} \to \mathrm{DH} \simeq \mathrm{Mod}_{\mathcal{H}_{(-)/\mathbb{C}}}(\mathrm{DM}_{\text{ét}}) \xrightarrow{\rho_H} \mathrm{Mod}_{\rho_H(\mathcal{H}_{(-)/\mathbb{C}})}(\mathrm{DH}) \to \mathrm{Mod}_{\mathbb{Q}}(\mathrm{DH}) \xleftarrow{\sim} \mathrm{DH}$$

where the map $\rho_H(\mathcal{H}_{(-)/\mathbb{C}}) \to \mathbb{Q}$ is induced by the co-unit of the adjunction (ρ_H, ρ_*^H).

The first observation is a consequence of the commutation with the operations:

Lemma 2.3. *For each finite type $\mathbb{C}$-scheme, the functor*

$$\underline{\rho_H}\colon \mathrm{DH} \to \mathrm{D}_H$$

is fully faithful. Moreover the map

$$\mathcal{H}_{X/\mathbb{C}} \to \mathcal{H}_X$$

is an equivalence.

Proof. The proof is the same as the proof of [Tub23, Lemma 4.14]. We recall it here. First over $\mathrm{Spec}\,\mathbb{C}$ the functor is fully faithful because all categories are rigid by [Dre18, Lemma 4.11]. Over a general base it suffices to prove that the functor is fully faithful on compact objects, and then the commutation with internal homomorphisms reduces to the case of $\mathrm{Spec}\,\mathbb{C}$. The claim about the map of algebras being an equivalence is a play of adjunctions. □

Denote by ${}^{\mathrm{ord}}\mathrm{H}^n$ the cohomology functor for the ordinary t-structure.

Definition 2.4 (Ayoub)**.** Let X be a finite type $\mathbb{C}$-scheme.

1. We let $\mathrm{MHM}_{\mathrm{geo}}(X)$ (*resp.* $\mathrm{MHM}_{hod}(X)$) be the full subcategory of $\mathrm{IndMHM}_{\mathrm{ord}}(X)$ generated under kernels, cokernels, extensions and filtered colimits by objects of the form ${}^{\mathrm{ord}}\mathrm{H}^n(f_*\pi_Y^*K)(m)$ with $K \in \mathrm{Mod}_{\mathbb{Q}}$ (*resp.* with $K \in \mathrm{D}_H(\mathbb{C})$), $f : Y \to X$ a proper morphism, $\pi_Y : Y \to \mathrm{Spec}\,\mathbb{C}$ the structural morphism and $n, m \in \mathbb{Z}$. These are Grothendieck abelian categories that we called Mixed Hodge modules of geometric origin (*resp.* of Hodge origin).

2. We let $\mathrm{D}_{H,\mathrm{geo}}(X)$ (*resp.* $\mathrm{D}_{H,\mathrm{hod}}(X)$) be the full subcategory of complexes $K \in \mathrm{D}_H(X)$ such that for all $n \in \mathbb{Z}$ we have ${}^{\mathrm{ord}}\mathrm{H}^n(K) \in \mathrm{MHM}_{\mathrm{geo}}(X)$ (*resp.* we have ${}^{\mathrm{ord}}\mathrm{H}^n(K) \in \mathrm{MHM}_{\mathrm{hod}}(X)$). Those are stable ∞-categories on which the canonical t-structure on $\mathrm{D}_H(X) = \mathrm{D}(\mathrm{IndMHM}_{\mathrm{ord}}(X))$ restricts. Moreover they are stable under pullbacks by proper base change.

Lemma 2.5. *The functors*

$$\underline{\rho_H}\colon \mathrm{DH} \to \mathrm{D}_H$$

and

$$\underline{\rho_{\mathbf{H}}}\colon \mathbf{DH} \to \mathrm{D}_H$$

land in $\mathrm{D}_{H,\mathrm{geo}}$ and $\mathrm{D}_{H,\mathrm{hod}}$.

Proof. This results from the commutation with the operations and the fact that $\mathrm{DH}(X)$ (*resp.* $\mathbf{DH}(X)$) is generated under colimits by the $p_*\pi_X^*K(m)$ for $K \in \mathrm{Mod}_{\mathbb{Q}}$ (*resp.* $K \in \mathrm{D}_H(\mathbb{C})$), thanks to [Ayo07, Lemme 2.2.23]. □

Proposition 2.6. *Let η be the generic point of an irreducible finite type $\mathbb{C}$-scheme, and consider the functor*

$$\mathrm{colim}_{\eta\in U}\mathrm{DH}(U) \to \mathrm{colim}_{\eta\in U}(\mathrm{D}_{H,\mathrm{geo}}(U))$$

induced by $\underline{\rho_H}$, where the colimit runs over all the open subsets of X that contain η. It is an equivalence.

Proof. By Lemma 2.3 this functor is fully faithful. It remains to show that it is essentially surjective. Also by continuity of motives [EHIK21, Lemma 5.1 (ii)] and [Lur22, Theorem 4.8.5.11], the left hand side is canonically equivalent to $\mathrm{DH}(\eta) := \mathrm{Mod}_{u^*\mathcal{H}_\mathbb{C}}(\mathrm{DM}_{\text{ét}}(\eta))$, where $u : \eta \to \mathrm{Spec}\,\mathbb{C}$ is the structural morphism. Moreover, by [Ayo07, Lemma 2.2.27], the ∞-category $\mathrm{DH}(\eta)$ is generated under colimits by the $f_*\pi_Y^*K(n)$ where $f : Y \to \eta$ is a smooth and projective morphism, $n \in \mathbb{Z}$ and $K \in \mathrm{Mod}_\mathbb{Q}$. By spreading out, such an object is the restriction to η of a $g_*\pi_Z^*K(n)$ with $f : Z \to U$ a smooth projective morphism. This means that there exists a finite category I, an open subset U and a functor $F : I \to \mathrm{DH}(U)$ such that for each $i \in I$ the object $F(i)$ is of the form $g^i_*\pi_{Z_i}K_i(n_i)[m_i]$ with $g^i : Z_i \to U$ projective and smooth, $K_i \in \mathrm{Mod}_\mathbb{Q}$ and $n_i, m_i \in \mathbb{Z}$, such that $(\mathrm{colim}_I F)_\eta = f_*\pi_Y^*K(n)$. Thus in the colimit

$$\mathcal{D} := \mathrm{colim}_{\eta\in U}(\mathrm{D}_{H,\mathrm{geo}}(U))$$

the object $f_*\pi_Y^*K(n)$ is the finite colimit of the image of F by $\underline{\rho_H}$. This proves that the category $\mathcal{D}$ is generated under colimits and truncations by images in the colimit of objects of the form $g_*\pi_Z^*K(n)$ with $g : Z \to U$ smooth and projective.

For each U, the compact objects of $\mathrm{D}_{H,\mathrm{geo}}(U)$ have an ordinary t-structure and its heart is small category defined in the same way as $\mathrm{MHM}^{\mathrm{ord}}_{\mathrm{geo}}(U)$, but only allowing only direct factors instead of all filtered colimits. As all transitions in the diagram are t-exact, the colimit of the compact objects (which are the compact objects of the colimit) has a bounded t-structure. It turns out that to prove that the functor of the proposition is essentially surjective, it suffices to check that it reaches all objects of the heart of the compact objects. Thanks to [Ayo22, Lemma 1.100], we see that it suffices to show that all subquotients of the image in the colimit of all ${}^{\mathrm{ord}}\mathrm{H}^n(f_*K)$, for $K \in \mathrm{Mod}_\mathbb{Q}$ compact and $f : Y \to U$ projective and smooth, are in the image. Now by dévissage we can assume K to be pure, so that by the conservation of weight by pushforwards by smooth proper maps, and semi-simplicity of pure objects we have ${}^{\mathrm{ord}}\mathrm{H}^n(f_*\pi_Y^*K)$, and each of its subquotients are direct factors of $f_*\pi_Y^*K$ thus lie in the image! This finishes the proof. □

Theorem 2.7. *Let X be a finite type $\mathbb{C}$-scheme. The functors*

$$\underline{\rho_H}\colon \mathrm{DH}(X) \to \mathrm{D}_{H,\mathrm{geo}}(X)$$

and

$$\underline{\rho_{\mathbf{H}}}\colon \mathbf{DH}(X) \to \mathrm{D}_{H,\mathrm{hod}}(X)$$

are equivalences of categories.

Proof. This is now a simple Noetherian induction: if $K \in \mathrm{D}_{H,\mathrm{geo}}(X)$, we may find a nonempty irreducible open subset U of X with generic point η and Proposition 2.6 ensures that up to reducing the size of U, the restriction $K_{|U}$ of K to U is in the image, but the localisation sequence

$$j_!K_{|U} \to K \to i_*i^*K$$

with $j : U \to X$ and $i : Z \to X$ the complement, together with the fully faithfulness and a Noetherian induction, give that K is in the image. □

Corollary 2.8. *All desiderata of Drew in [Dre18] are fulfilled.*

Proof. Only was missing the t-structure, which we now have, and the compatibility with the weight structure, which is clear when looking at the pure objects. □

Remark 2.9. In a forthcoming work with Raphaël Ruimy, we use this point of view of motivic Hodge modules to construct categories of mixed Hodge modules of geometric origin that have $\mathbb{Z}$-linear coefficients.

Although Corollary 2.8 ensures that the goals of Drew are achieved, it may still seem unsatisfactory: another motivation for Drew's work was probably to give an alternative construction of mixed Hodge modules. Here our proof uses mixed Hodge modules hence it is not quite right. It should be possible to prove that motivic Hodge modules afford a t-structure without comparing them to M. Saito's mixed Hodge modules.

We give here a weak reason for which one could possibly define the t-structure only by geometric means (a t-structure on the generic points is enough to have a t-structure for every variety by gluing, see [Bon15, Theorem 3.1.4]):

Proposition 2.10. *Let $K = \mathbb{C}(X)$ be the fraction field of an integral complex algebraic variety. Then the t-structure on*

$$\mathrm{DH}(K/\mathbb{C}) := \mathrm{colim}_U \mathrm{DH}(U),$$

where the colimit runs over nonempty open subsets of X, is characterized by the fact that $\mathrm{DH}(K/\mathbb{C})^{\leqslant 0}$ is the smallest subcategory of $\mathrm{DH}(K/\mathbb{C})$ stable under colimits, twists and extensions that contains the objects $f_\sharp \mathbb{Q}_X$ for $f : X \to \operatorname{Spec} K$ smooth and affine.

Proof. We have that $\mathrm{DH}(K/\mathbb{C})$ is compactly generated by $\mathrm{colim}_U \mathrm{DH}_c(U)$ where this time the colimit is taken in Cat_∞. Thus we know that $\mathrm{DH}(K/\mathbb{C})^{\leqslant 0}$ is the prestable ∞-category

$$\mathrm{DH}(K/\mathbb{C})^{\leqslant 0} \simeq \mathrm{Ind}\big(\mathrm{colim}_U \mathrm{DH}_c(U)^{\leqslant 0}\big).$$

In particular, as the heart of each $\mathrm{DH}(U)$ is generated under kernels, cokernels, extensions, twists and filtered colimits by objects of the form ${}^{\mathrm{ord}}\mathrm{H}^n(f_*\mathbb{Q}_Y)$ with $f\colon Y \to U$ proper, the proper base change theorem ensures that the heart of $\mathrm{DH}(K/\mathbb{C})$ is generated under the same operations by the ${}^{\mathrm{ord}}\mathrm{H}^n(f_*\mathbb{Q}_Y)$ with $f : Y \to \operatorname{Spec} K$ proper and $n \in \mathbb{N}$. In fact we have seen in the proof of Proposition 2.6 that we can even assume that f is smooth and projective. Thus the prestable ∞-category $\mathrm{DH}(K/\mathbb{C})^{\leqslant 0}$ is generated under colimits, extensions and twists by objects of the form ${}^{\mathrm{ord}}\mathrm{H}^n(f_*\mathbb{Q})$ for $f\colon Y \to \operatorname{Spec} K$ smooth and projective. Moreover as this is true over a small open subset U of X, we have a decomposition in $\mathrm{DH}(K/\mathbb{C})$ of the form

$$f_*\mathbb{Q}_Y \simeq \bigoplus_n {}^{\mathrm{ord}}\mathrm{H}^n(f_*\mathbb{Q}_Y)[-n].$$

This implies that $\mathrm{DH}(K/\mathbb{C})^{\leqslant 0}$ is generated by the $f_*\mathbb{Q}_Y[2\dim Y]$ for $f : Y \to \operatorname{Spec} K$ projective and smooth. By covering Y with smooth and affine schemes $(U_i)_{i\leqslant n}$, if one denote by $U_J = \cap_{i\in J} U_J$ for $J \subset \{1, \ldots, n\}$, as we have a colimit diagram

$$\cdots \;\substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow}\; \bigoplus\nolimits_{|J|=2} f_!^{U_J}\mathbb{Q}_{U_J}[2d] \;\substack{\longleftarrow\\\longrightarrow\\\longleftarrow}\; \bigoplus\nolimits_{|J|=1} f_!^{U_J}\mathbb{Q}_{U_J}[2d] \longrightarrow f_*\mathbb{Q}_Y[2d]$$

by Zariski descent, with $f^{U_J} : U_J \to Y \to \operatorname{Spec} K$ and $d = \dim Y$, we see that $\mathrm{DH}(K/\mathbb{C})^{\leqslant 0}$ is generated by the $f_!\mathbb{Q}_X[2d]$ for $f : X \to \operatorname{Spec} K$ smooth and affine of dimension d. As $f_\sharp\mathbb{Q}_X \simeq f_!\mathbb{Q}_X(d)[2d]$, this finishes the proof.

□

3 Extension of the derived category of mixed Hodge modules to algebraic stacks.

3.1 Extensions and operations.

We have at hand a h-hypersheaf

$$\mathrm{D_H}\colon \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L).$$

The functor D_H extends (as a right Kan extension) canonically to a h-hypersheaf all Artin stacks over $\mathbb{C}$. More explicitly, given $\pi : X \to \mathfrak{X}$ a presentation of an Artin stack $\mathfrak{X}$, so that π is smooth and X is a scheme, and denoting by

$$X^{n/\mathfrak{X}} := X \times_{\mathfrak{X}} X \times_{\mathfrak{X}} \cdots X$$

the n-th fold of X over $\mathfrak{X}$ (this is a scheme as the diagonal of $\mathfrak{X}$ is representable), we have a limit diagram in Pr^L

$$\mathrm{D}_H(\mathfrak{X}) \xrightarrow{\pi^*} \mathrm{D}_H(X) \rightrightarrows \mathrm{D}_H(X \times_{\mathfrak{X}} X) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots \,. \tag{3.0.1}$$

Using Liu and Zheng gluing technique ([LZ17]) as Khan in [Kha19, Appendix A.], one can prove that the extension of D_H to (higher) Artin stacks still have the six operations. This subsection is more or less book keeping of the work of Liu, Zheng and Khan. More precisely, we have:

Theorem 3.1 (Liu-Zheng, Khan)**.**

1. *For every Artin stack $\mathfrak{X}$ there is a closed symmetric monoidal structure on $\mathrm{D}_H(\mathfrak{X})$.*
2. *For any morphism $f : \mathfrak{Y} \to \mathfrak{X}$ there is an adjunction*

$$\mathrm{D}_H(\mathfrak{X}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{D}_H(\mathfrak{Y}) \quad (f^* \dashv f_*)$$

   with f^* a symmetric monoidal functor.

3. *For any locally of finite type morphism of Artin stacks $f : \mathfrak{X} \to \mathfrak{Y}$ there is an adjunction (functorial in f)*

$$\mathrm{D}_H(\mathfrak{X}) \underset{f^!}{\overset{f_!}{\rightleftarrows}} \mathrm{D}_H(\mathfrak{Y}) \quad (f_! \dashv f^!)\,.$$

4. *The operations $f_!$ satisfy base change and projection formula against g^*, and the operations $f^!$ satisfy base change against g_*. If f is representable by Deligne-Mumford stacks, then there is a natural transformation $\alpha_f : f_! \to f_*$, which is an isomorphism if f is proper and representable by algebraic spaces.*
5. *For any closed immersion $i : \mathfrak{Z} \to \mathfrak{X}$ of Artin stacks with $j : \mathfrak{U} \to \mathfrak{X}$ the inclusion of the open complement, we have a pullback diagram*

$$\begin{array}{ccc} \mathrm{D}_H(\mathfrak{Z}) & \xrightarrow{i_*} & \mathrm{D}_H(\mathfrak{X}) \\ \downarrow & & \downarrow{\scriptstyle j^*} \\ * & \longrightarrow & \mathrm{D}_H(\mathfrak{U}) \end{array}\,.$$

6. *For any Artin stack $\mathfrak{X}$, the functor*

$$\pi_* : \mathrm{D}_H(\mathbb{A}^1_{\mathfrak{X}}) \to \mathrm{D}_H(\mathfrak{X})$$

is fully faithful.

7. *For a smooth morphism $f : \mathfrak{Y} \to \mathfrak{X}$, if one denotes by $\Sigma^{\mathcal{L}_{\mathfrak{Y}/\mathfrak{X}}} : \mathrm{D}_H(\mathfrak{Y}) \to \mathrm{D}_H(\mathfrak{Y})$ the endofunctor $p_\sharp s_*$ with s the zero section of the vector bundle $p : \mathbb{V}(\mathcal{L}_{\mathfrak{Y}/\mathfrak{X}}) \to \mathfrak{Y}$ associated to the relative differentials of f (in general, one would have to consider the cotangent complex, but our stacks are classical), we have a purity isomorphism*

$$\mathfrak{p}_f : \Sigma^{\mathcal{L}_{\mathfrak{Y}/\mathfrak{X}}} \circ f^* \simeq f^!.$$

8. *The $!$-functoriality $\mathrm{D}_H(-)^!$ is also an étale hypersheaf.*

Proof. The proofs of [Kha19, Appendix A] hold for any motivic coefficient system, so that 1,2,3 and 4 are [Kha19, Theorem A.5], 5. is [Kha19, Theorem A.9], 6. is [Kha19, Proposition A.10] and 7. is [Kha19, Theorem A.13]. The last point 8. follows from [LZ17, Proposition 4.3.5]. □

Remark 3.2. All results above hold more generally for higher Artin stacks, except for 4. where one need f to be 0-truncated on top of being proper for $f_!$ to be isomorphic to f_*. Moreover, for a smooth morphism of finite type, we can choose a trivialization of the cotangent complex $\mathcal{L}_{\mathfrak{Y}/\mathfrak{X}}$ so that it is only a Tate twist and a shift, as usual when dealing with orientable theories.

Remark 3.3. Is is clear that the operations preserve the Kan extension of mixed Hodge modules of geometric or Hodge origin to stacks, and all the results we stated above and below also hold in this more restrictive setting. We choose to not mention this at every theorem for lisibility.

Now, as $\mathrm{D}^b(\mathrm{MHM}(-))$ also satisfies h-descent, we can also right Kan extend it and have the same formula as (3.0.1). In particular as limits in Cat_∞ are computed term wise, if one denote by $\mathrm{D}^b_H(-)$ the extension of $\mathrm{D}^b(\mathrm{MHM}(-))$ to (higher) Artin stacks, we have a fully faithful natural transformation

$$\iota \colon \mathrm{D}^b_H(-) \to \mathrm{D}_H(-).$$

The question whether for a general morphism f of Artin stacks, the operations f_* and $f_!$ preserve constructibility is subtle. Of course, it is false in general that $f_*\mathbb{Q}$ is cohomologically bounded.

For a finite type $\mathbb{C}$-scheme X, the category $\mathrm{D}_H(X) = \mathrm{Ind}\mathrm{D}^b(\mathrm{MHM}(X))$ admits a t-structure by [Lur, Lemma C.2.4.3]. In fact, as $\mathrm{MHM}(\mathrm{Spec}\,\mathbb{C}) = \mathrm{MHS}^P_{\mathbb{C}}$ is of cohomological dimension 1 ([Bei86]), it is not hard to so that $\mathrm{MHM}(X)$ is of finite cohomological dimension (because the category of perverse sheaves is), so that the canonical functor $\mathrm{D}_H(X) \to \mathrm{D}(\mathrm{IndMHM}(X))$ is an equivalence. In particular (see [Lur, C.5.4.11]) the t-structure on $\mathrm{D}_H(X)$ is right complete, left separated and compatible with filtered colimits.

Proposition 3.4. *Let $\mathfrak{X}$ be an Artin stack locally of finite type over $\mathbb{C}$. The ∞-category $\mathrm{D}_H(\mathfrak{X})$ admits a t-structure such that for any presentation $\pi : X \to \mathfrak{X}$ the conservative functor π^* is t-exact up to a shift. Moreover, the t-structure is right complete, left separated and compatible with filtered colimits. It restricts to a t-structure on $\mathrm{D}^b_H(-)$.*

Proof. We can assume that $\mathfrak{X}$ is connected. Choose a presentation $\pi : X \to \mathfrak{X}$ which is smooth of relative dimension d for some integer d. Then all the projections

$$p_n : X \times_{\mathfrak{X}} X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X \to X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$$

are also smooth of relative dimension d, so that each $p_n^*[d]$ are t-exact for the perverse t-structure. As (3.0.1) can be done with the shifts $p_n[d]$, this creates a t-structure on the limit of the Čech nerve of π such that π^* is t-exact. It does not depend on π as one can see by taking another presentation $p : Y \to \mathfrak{X}$ and then $Y \times_{\mathfrak{X}} X \to \mathfrak{X}$. As π^* is conservative, the t-structure on $\mathrm{D}_H(\mathfrak{X})$ is left separated and compatible with colimits. It obviously restricts to $\mathrm{D}^b_H(\mathfrak{X})$. For the right completeness, we want to show that the natural functor

$$\mathrm{D}_H(\mathfrak{X}) \to \lim\Big(\cdots \xrightarrow{\tau^{\geqslant 1}} \mathrm{D}_H^{\geqslant 1}(\mathfrak{X}) \xrightarrow{\tau^{\geqslant 2}} \mathrm{D}_H^{\geqslant 2}(\mathfrak{X}) \xrightarrow{\tau^{\geqslant 3}} \mathrm{D}_H^{\geqslant 3}(\mathfrak{X}) \xrightarrow{\tau^{\geqslant 4}} \cdots\Big)$$

is an equivalence. As (shifts of) pullbacks by smooth maps are t-exact, this can be checked locally on $\mathfrak{X}$, hence holds. □

Definition 3.5. Let $\mathfrak{X}$ be an Artin stack locally of finite type over $\mathbb{C}$. The ∞-category of cohomologically constructible mixed Hodge modules over $\mathfrak{X}$ is

$$\mathrm{D}_{H,c}(\mathfrak{X}) := \{K \in \mathrm{D}_H(\mathfrak{X}) \mid \forall n \in \mathbb{Z}, \mathrm{H}^n(K) \in \mathrm{D}^b_H(\mathfrak{X})\}.$$

By definition the t-structure restricts to $\mathrm{D}_{H,c}(\mathfrak{X})$, and the heart is the same as the heart of $\mathrm{D}^b_H(\mathfrak{X})$.

For a finite type $\mathbb{C}$-scheme X, the stable ∞-category $\mathrm{D}^b(\mathrm{MHM}(X))$ also have a *ordinary t-structure*, and in [Tub23] (where the t-structure had the unfortunate name ‘constructible’) we proved that it is the derived category of the constructible heart. In fact, all the considerations above about the perverse t-structure are true for the constructible t-structure. We mention them because the ordinary t-structure will happen to be handy when proving the constructibility of the operations.

Proposition 3.6. *Let $\mathfrak{X}$ be an Artin stack over $\mathbb{C}$. The ∞-category $\mathrm{D}_H(\mathfrak{X})$ admits a ordinary t-structure such that for any presentation $\pi : X \to \mathfrak{X}$ the conservative functor π^* is t-exact if we endow $\mathrm{D}_H(X)$ with the t-structure induced by the canonical t-structure on $\mathrm{D}^b(\mathrm{MHM}_{\mathrm{ord}})$. Moreover, the t-structure is right complete, left separated and compatible with filtered colimits. It restricts to a t-structure on $\mathrm{D}^b_H(-)$. All pullbacks by all morphisms of Artin stacks are t-exact for this t-structure.*

Proof. The proof is exactly the same as for Proposition 3.4, except that we do not need to shift the pullback functors. □

We will denote by ${}^{\mathrm{ord}}\mathrm{H}^n$ the cohomology objects for the ordinary t-structure.

Proposition 3.7. *Let $\mathfrak{X}$ be an Artin stack locally of finite type over $\mathbb{C}$ and let $K \in \mathrm{D}_H(\mathfrak{X})$. Then $K \in \mathrm{D}_{H,c}(\mathfrak{X})$ if and only if for all $n \in \mathbb{Z}$ the object ${}^{\mathrm{ord}}\mathrm{H}^n(K)$ is in $\mathrm{D}^b_H(\mathfrak{X})$.*

Proof. Both conditions on K are local on $\mathfrak{X}$, hence we can assume that $\mathfrak{X}$ is a finite type $\mathbb{C}$scheme X. If K is bounded the result is trivial, and if not, we can reduce to the bounded case by noting the following: $\tau^{p\leqslant n}$ preserves $\mathrm{D}_H(X)^{nat\leqslant n}$, $\tau^{nat\geqslant n}$ preserves $\mathrm{D}_H(X)^{p\geqslant 0}$, the restriction of $\tau^{p\geqslant n}$ to $\mathrm{D}_H(X)^{nat\geqslant n}$ is the identity and the restriction of $\tau^{nat\leqslant n}$ to $\mathrm{D}_H(X)^{p\leqslant n}$ is the identity. □

Theorem 3.8 (Liu-Zheng)**.** *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of Artin stacks locally of finite type over $\mathbb{C}$. Then we have the following:*

1. *The ∞-category $\mathrm{D}^-_{H,c}(\mathfrak{X})$ is stable under tensor products, and f^* restricts to a functor*

$$f^* \colon \mathrm{D}_{H,c}(\mathfrak{X}) \to \mathrm{D}_{H,c}(\mathfrak{Y}).$$

2. *The functor f_* restricts to a functor*

$$f_*\colon \mathrm{D}^+_{H,c}(\mathfrak{Y}) \to \mathrm{D}^+_{H,c}(\mathfrak{Y})$$

and even to

$$f_*\colon \mathrm{D}_{H,c}(\mathfrak{Y}) \to \mathrm{D}_{H,c}(\mathfrak{Y})$$

if f is representable by algebraic spaces.

3. *The functor $f_!$ restricts to a functor*

$$f_!\colon \mathrm{D}^-_{H,c}(\mathfrak{Y}) \to \mathrm{D}^-_{H,c}(\mathfrak{Y})$$

and even to

$$f_!\colon \mathrm{D}_{H,c}(\mathfrak{Y}) \to \mathrm{D}_{H,c}(\mathfrak{Y})$$

if f is representable by algebraic spaces.

4. *The functor $f^!$ restricts to a functor*

$$f^!\colon \mathrm{D}_{H,c}(\mathfrak{X}) \to \mathrm{D}_{H,c}(\mathfrak{Y}).$$

5. *The internal Hom functor restricts to a functor*

$$\mathrm{D}^-_{H,c}(\mathfrak{X})^{\mathrm{op}} \times \mathrm{D}^+_{H,c}(\mathfrak{X}) \to \mathrm{D}^+_{H,c}(\mathfrak{X}).$$

Proof. The proof is essentially the same as the proofs of [LZ17, 6.4.4 and 6.4.5]. The point 1. is easy. We show how to obtain 2. for example: Take $\pi : X \to \mathfrak{X}$ a presentation of $\mathfrak{X}$ and form the pullback

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{g} & X \\ \downarrow{\scriptstyle q} & \lrcorner & \downarrow{\scriptstyle \pi} \\ \mathfrak{Y} & \xrightarrow{f} & \mathfrak{X} \end{array}.$$

Then to show that each ${}^{\mathrm{ord}}\mathrm{H}^n(f_!K)$ are constructible, we may check it locally, whence as π^* is t-exact it suffices to show that ${}^{\mathrm{ord}}\mathrm{H}^n(\pi^* f_!K)$ is constructible. By base change, it suffices to show that ${}^{\mathrm{ord}}\mathrm{H}^n(g_!q^*K)$ is constructible: we reduced to the case where $\mathfrak{X}$ is a scheme. Now, choose a presentation $r : Z \to \mathfrak{Z}$ of $\mathfrak{Z}$. Descent for !-functoriality implies that $q^*K = \mathrm{colim}_\Delta (g_n)_! r_n^! q^*K$ where (g_n) is the composition of the Cech nerve (r_n) of r with g. The spectral sequence induced by this geometric realisation reads

$$E_1^{p,q} = {}^{\mathrm{ord}}\mathrm{H}^{-q}((g_p)_! r_p^! q^*K) \Rightarrow {}^{\mathrm{ord}}\mathrm{H}^{p-q}(g_! q^*K).$$

Now the each $r_p^! q^*K$ is cohomologically constructible by purity (7. of Theorem 3.1) and each g_p is a morphism of schemes, of relative dimension the same relative dimension as g. In particular, all g_p have the same cohomological amplitude thus our spectral sequence is concentrated in a shifted quadrant: it vanishes if $p < 0$ and if q is smaller than some bound depending on the cohomological amplitude of $g_!$ and of K. This implies that the spectral sequence collapse and we have that $\mathrm{H}^n(g_! q^*K)$ is constructible for all n. In the case f is representable by algebraic spaces, we can reduce to the case of schemes where it follows from finite cohomological amplitude. The case of f_* is similar. □

Remark 3.9. We may have to use the same extension to stack for others systems of coefficients such as analytic sheaves or étale motives. Because the functor between those and mixed Hodge modules commute with all the operations on schemes, this is also the case for their extensions to stacks.

3.2 Duality

We want a working Verdier duality on algebraic stacks locally of finite type over the complex numbers.

Definition 3.10. Let $\mathfrak{X}$ be a algebraic stack locally of finite type over the complex numbers. The dualizing object on $\mathfrak{X}$ is the object $\pi_{\mathfrak{X}}^!\mathbb{Q}(0) \in \mathrm{D}_H(\mathfrak{X})$ with $\pi_{\mathfrak{X}} \colon \mathfrak{X} \to \operatorname{Spec}\mathbb{C}$. The Verdier duality functor is

$$\mathbb{D}_{\mathfrak{X}} := \mathscr{H}om(-, \omega_{\mathfrak{X}}).$$

If follows directly from the projection formula between $g_!$ and g^*, for g a morphism of algebraic stacks, that there are natural equivalences

$$g^! \circ \mathbb{D}_{\mathfrak{X}} \simeq \mathbb{D}_{\mathfrak{Y}} \circ g^*$$

and

$$g_* \circ \mathbb{D}_{\mathfrak{Y}} \simeq \mathbb{D}_{\mathfrak{X}} \circ g_!$$

for any morphism $g : \mathfrak{Y} \to \mathfrak{X}$ between algebraic stacks locally of finite type over $\mathbb{C}$.

Lemma 3.11. *Let $g : \mathfrak{Y} \to \mathfrak{X}$ be a* smooth *morphism between algebraic stacks locally of finite type over $\mathbb{C}$. Then there is also a natural equivalence*

$$g^* \circ \mathbb{D}_{\mathfrak{X}} \simeq \mathbb{D}_{\mathfrak{Y}} \circ g^!$$

of functors $\mathrm{D}_H(\mathfrak{X}) \to \mathrm{D}_H(\mathfrak{Y})$.

Proof. As if g is étale we have $g^* \simeq g^!$ thus the lemma holds, we can check this étale locally on $\mathfrak{X}$, thus we may assume that g is of relative dimension d. By purity and the smooth/closed projection formulae, there is an isomorphism $\Sigma^{\mathcal{L}_g}(\mathbb{Q}) \otimes g^* \simeq g^!$. As g is smooth, it is formally smooth hence the cotangent complex $\mathcal{L}_g$ of g is locally free and of finite rang. Thus up to take a covering we have $\mathcal{L}_g \simeq \mathscr{O}_{\mathfrak{X}}^d$, so that $\Sigma^{\mathcal{L}_g}(\mathbb{Q}) \simeq \mathbb{Q}(d)[2d]$. As Tate twists and shifts commute with all operations (up to a sign), the lemma follows from the identity $g^! \circ \mathbb{D}_{\mathfrak{X}} \simeq \mathbb{D}_{\mathfrak{Y}} \circ g^*$ above. ☐

Proposition 3.12. *Verdier duality is an anti auto-equivalence when restricted to $\mathrm{D}^b_H(\mathfrak{X})$. It swaps $!$ and $*$ when they preserve D^b_H and it is perverse t-exact.*

Proof. First note that duality preserves D^b_H because this can be checked locally. There is a canonical map

$$\mathrm{Id} \to \mathbb{D}_{\mathfrak{X}} \circ \mathbb{D}_{\mathfrak{X}}.$$

For a given $M \in \mathrm{D}^b_H(\mathfrak{X})$ and a presentation $g : \mathfrak{X} \to X$, it suffices to check that the map

$$g^*M \to g^*(\mathbb{D}_{\mathfrak{X}} \circ \mathbb{D}_{\mathfrak{X}}(M))$$

is an equivalence. But now by Lemma 3.11 the above map is an equivalence because it is the map

$$g^*M \to \mathbb{D}_X \circ \mathbb{D}_X(g^*M).$$

The remaining assertions follows from the autoduality. ☐

Corollary 3.13. *Verdier duality in fact extends to an auto-equivalence of $\mathrm{D}_{H,c}(\mathfrak{X})$. It swaps $!$ and $*$.*

Proof. By the previous assertion, the functor $\mathbb{D}_{\mathfrak{X}}$ is perverse t-exact, and an isomorphism on the heart. ☐

Proposition 3.14. *All result above about the extension of the operations work for the category* $\mathrm{D}_B := \mathrm{Ind}\mathrm{D}^b_c((-)^{\mathrm{an}}, \mathbb{Q})$ *of Ind-constructible complexes, the forgetful functor* $\mathrm{rat}\colon \mathrm{D}_H \to \mathrm{D}_B$ *extends to schemes, and is compatible with all the operations above.*

Proof. This follows from the fact that the proposition holds over finite type $\mathbb{C}$-schemes. ☐

3.3 Weights

Let $w \in \mathbb{Z}$ and X be a finite type $\mathbb{C}$-scheme. Recall that following [Mor08] we can construct a weight-t-structure on $\mathrm{D}^b_H(X)$ by setting

$$\mathrm{D}^b_H(X)^{\omega \leqslant w} := \{K \in \mathrm{D}^b_H(X) \mid \forall n \in \mathbb{Z},\ {}^{\mathrm{p}}\mathrm{H}^n(K) \text{ is of weights } \leqslant w\}$$

and

$$\mathrm{D}^b_H(X)^{\omega \geqslant w} := \{K \in \mathrm{D}^b_H(X) \mid \forall n \in \mathbb{Z},\ {}^{\mathrm{p}}\mathrm{H}^n(K) \text{ is of weights } \geqslant w\}$$

This form a t-structure (of trivial heart) on $\mathrm{D}^b_H(X)$, such that the truncations functors are exact functors, t-exact for the perverse t-structure. They give a filtration $(\omega^{\leqslant w} K)_w$ on each complex K, that give back the weight filtration on the heart. Of course, for any map $f : Y \to X$ of finite type $\mathbb{C}$-schemes, and any integer $d \in \mathbb{Z}$, the functor $f^*[d]$ is t-exact for the weight t-structures. This induces a weight t-structure on $\mathrm{D}_H(X)$ by indization and it has a similar description.

Proposition 3.15. *Let $\mathfrak{X}$ be an Artin stack locally of finite type over $\mathbb{C}$. Then the ∞-categories $\mathrm{D}_H(\mathfrak{X})$ and $\mathrm{D}_{H,c}(\mathfrak{X})$ admit weight filtrations such that for every map $f : X \to \mathfrak{X}$ with X a finite type $\mathbb{C}$-scheme and every integer d, the functor f^* preserves the weight filtration.*

Proof. Once again assume that $\mathfrak{X}$ is connected, choose $\pi : X \to \mathfrak{X}$ a presentation of $\mathfrak{X}$, of relative dimension d. Then if f_n is a part of the Cech nerve of π, the functors $f_n^*[d]$ are t-exact for the weight t-structures, thus this induces a weight t-structure on the limit as in (3.0.1). Of course it restricts to $\mathrm{D}^b_H(\mathfrak{X})$, and then to $\mathrm{D}_{H,c}(\mathfrak{X})$. ☐

Definition 3.16. We say that an object $K \in \mathrm{D}_{H,c}(\mathfrak{X})$ is of weights $\leqslant w$ if for all $n \in \mathbb{Z}$ we have ${}^{\mathrm{ord}}\mathrm{H}^n(K)$ is of weights $\leqslant w + n$ (and similarly for $\geqslant w$). We shall denote by $\mathrm{D}_{H,c}(\mathfrak{X})_{\leqslant w}$ and $\mathrm{D}_{H,c}(\mathfrak{X})_{\geqslant w}$ the corresponding full subcategories.

Remark 3.17. The above definition *differs* from the weight t-structure we used to define weights.

Proposition 3.18. *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of algebraic stacks locally of finite type over $\mathbb{C}$ and let $w, w' \in \mathbb{Z}$.*

1. *Verdier duality on $\mathfrak{X}$ swaps $\mathrm{D}_{H,c}(\mathfrak{X})_{\leqslant w}$ and $\mathrm{D}_{H,c}(\mathfrak{X})_{\geqslant w}$.*
2. *The pullback f^* sends $\mathrm{D}_{H,c}(\mathfrak{X})_{\leqslant w}$ to $\mathrm{D}_{H,c}(\mathfrak{Y})_{\leqslant w}$ and the exceptional pullback $f^!$ sends $\mathrm{D}_{H,c}(\mathfrak{X})_{\geqslant w}$ to $\mathrm{D}_{H,c}(\mathfrak{Y})_{\geqslant w}$. If f is smooth, the pullback functor f^* preserves weights.*
3. *The tensor product restricts to*

$$- \otimes - \colon \mathrm{D}^-_{H,c}(\mathfrak{X})_{\leqslant w} \times \mathrm{D}^-_{H,c}(\mathfrak{X})_{\leqslant w'} \to \mathrm{D}^-_{H,c}(\mathfrak{X})_{\leqslant w+w'}.$$

4. The internal homomorphism functor restricts to

$$\mathscr{H}om(-,-)\colon \mathrm{D}^{-}_{H,c}(\mathfrak{X})_{\leqslant w} \times \mathrm{D}^{-}_{H,c}(\mathfrak{X})_{\geqslant w'} \to \mathrm{D}^{-}_{H,c}(\mathfrak{X})_{\geqslant w'-w}.$$

Proof. By definition, if $\pi : X \to \mathfrak{X}$ is a presentation, the functor π^* is conservative and detects weights. Thus all results follows from the usual results over schemes. □

Proposition 3.19. *Let $\mathfrak{X}$ be an algebraic stack with affine stabilizers, locally of finite type over $\mathbb{C}$. Let $f : \mathfrak{X} \to \operatorname{Spec}\mathbb{C}$ be the structure map. If K is an object of $\mathcal{D}^b_H(\mathfrak{X})$ of weights $\leqslant w$, then $f_!K$ has weights $\leqslant w$.*

Proof. The main reduction of the proof is the same as in [Sun12].

By dévissage we can assume that K is pure of weight 0, and that it is a perverse sheaf. For each nonempty open subset $j : \mathfrak{U} \to \mathfrak{X}$ of reduced closed complement $i : \mathfrak{Z} \to \mathfrak{X}$, we have the following cofiber sequence:

$$j_!j^*K \to K \to i_*i^*K.$$

By Noetherian induction, it suffices to prove that $f_!(j_!j^*K)$ is of weight $\leqslant 0$. Moreover weights can be tested on an étale covering. In particular we can replace $\mathfrak{X}$ by any étale covering of any nonempty open subset $\mathfrak{U}$. Thus we can assume that K is lisse with constant Betti realisation. Let $\mathfrak{I}$ be the inertia stack of $\mathfrak{X}$. Recall that this is an algebraic group space defined by the pullback:

$$\begin{array}{ccc} \mathfrak{I} & \longrightarrow & \mathfrak{X} \\ \downarrow & \lrcorner & \downarrow \Delta \\ \mathfrak{X} & \xrightarrow[\Delta]{} & \mathfrak{X} \times \mathfrak{X} \end{array}$$

Let $\mathfrak{U}$ be the open substack of $\mathfrak{X}$ over which the map $\mathfrak{I} \to \mathfrak{X}$ is flat (this exists thanks to [Beh03, Proposition 4.14]). Then as in [Ols08, Section 1.5] we can form the rigidification X of $\mathfrak{U}$ with respect to $\mathfrak{I}$. It will be an algebraic stack with no nontrivial isomorphisms: an algebraic space. There is a faithfully flat map $\mathfrak{X} \to X$ with X a algebraic space. Let $U \subset X$ be a dense open subscheme. We replace $\mathfrak{X}$ by the inverse image of U in $\mathfrak{X}$.

We are in the following situation: The map $f : \mathfrak{X} \to \operatorname{Spec}\mathbb{C}$ factors through $\pi : \mathfrak{X} \to X$ and $g : X \to \operatorname{Spec}\mathbb{C}$, with π faithfully flat, and X a scheme. We want to compute the weights of $f_!K$ with K a lisse (with constant Betti realisation) perverse sheaf of pure weight 0 on $\mathfrak{X}$. We have the following Leray spectral sequence :

$$E_2^{p,q} = {}^{\mathrm{p}}\mathrm{H}^q(g_!\, {}^{\mathrm{p}}\mathrm{H}^p(\pi_!K)) \Rightarrow {}^{\mathrm{p}}\mathrm{H}^{p+q}(f_!K).$$

As we know that $g_!$ sends objects of negative weights to objects of negative weights, it suffices to show that each ${}^{\mathrm{p}}\mathrm{H}^p(\pi_!K)$ has weights $\geqslant q$. This can be tested on stalks because X is a scheme. Let x be a point of X and let y be a lift of x in $\mathfrak{X}$. Then by construction of the rigidification, the fiber of π at x is $B\mathrm{Aut}_y$ (see [Ols08, remark 1.5.4]): we have the following pullback square

$$\begin{array}{ccc} B\mathrm{Aut}_y & \xrightarrow{\iota} & \mathfrak{X} \\ \downarrow p & \lrcorner & \downarrow \pi \\ x & \longrightarrow & X \end{array}.$$

Proper base change gives that

$$(\pi_!K)_x \simeq p_!\iota^*K$$

hence we reduced to the case where $\mathfrak{X}$ is BG for G a linear algebraic group. In this case we follow the same proof as [Del74, Theorème 9.1.1]: take $T \subset G^0$ a maximal torus inside the neutral component G^0 of G. As it can be checked on the underlying $\mathbb{Q}$-structure, we have by the splitting principle, for every $M \in \mathrm{MHM}(BG)$ whose $\mathbb{Q}$-structure is constant, a monomorphism

$$\mathrm{H}^n(\pi_* M) \to \mathrm{H}^n(\delta_* i^* M)$$

with $\delta : BT \to \operatorname{Spec}\mathbb{C}$ and $i : BT \to BG$. Thus if we know that the weights of $\mathrm{H}^n(\delta_* i^* M)$ are positive, the same is true for $\mathrm{H}^n(\pi_* M)$, and applied to $M = \mathbb{D}(K)$ this gives finishes the proof of the proposition. Thus we are now in the following situation:

The object $M \in \mathrm{MHM}(BT)$ has constant $\mathbb{Q}$-structure, and we want to compute the weights of $\mathrm{H}^n(\pi_* M)$ where $\pi : BT \to \operatorname{Spec}\mathbb{C}$ is the structural map of the classifying space of a torus. Now as the Kunneth formula holds, it suffices to deal with the case of $T = \mathbb{G}_m$.

Moreover for an integer N map $f : \mathbb{P}^N_{\mathbb{C}} \simeq P/\mathbb{G}_m \to B\mathbb{G}_m$ with P the $\mathbb{G}_m$-bundle $\mathbb{P}(\mathscr{O}_{\mathbb{P}^N_{\mathbb{C}}}(1))$ over $\mathbb{P}^N_{\mathbb{C}}$ induces a map

$$\mathrm{H}^n(\pi_* M) \to \mathrm{H}^n((\pi_{\mathbb{P}^N_{\mathbb{C}}})_* f^* M)$$

which is a isomorphism if $n \leqslant 2N$, thus $\mathrm{H}^n(\pi_* M)$ is pure of weight $n/2$ for if n is even, and is the zero object otherwise. □

Corollary 3.20. *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of algebraic stacks locally of finite type over $\mathbb{C}$. Assume that $\mathfrak{Y}$ has affine stabilizers. Then if $K \in \mathrm{D}^b_H(\mathfrak{Y})$ is of weights $\leqslant w$ (*resp. *$\geqslant w$) the object $f_! K$ (*resp. *$f_* K$) also has weights $\leqslant w$ (*resp. *$\geqslant w$).*

Proof. By duality it suffices to deal with $f_!$. Let $\pi : X \to \mathfrak{X}$ be a presentation. Then it suffices to show that $\pi^* f_! K$ has weights $\leqslant w$, thus it suffices to check, for each point x of X, that the restriction to x of $f_! K$ has weights $\leqslant w$. But then the above proposition, combined with the $(x^*, f_!)$ base change, gives the result. □

Corollary 3.21. *Let $\mathfrak{X}$ be an algebraic stack with affine stabilizers, locally of finite type over $\mathbb{C}$. Then there is a weight structure* a la *Bondarko on $\mathrm{D}^b_H(\mathfrak{X})$ whose heart consist of objects of weight 0 as in Definition 3.16.*

Proof. We want to use [Bon10, Theorem 4.3.2] to construct a weight structure from its heart of pure objects of weight 0. Thus we want to prove that pure objects of weight 0 generated $\mathrm{D}^b_H(\mathfrak{X})$ as a thick subcategory, and that it is negative (for us, semi-simple will be enough). Thus if $K, L \in \mathrm{D}^b_H(\mathfrak{X})$ are pure of weight zero, we want to show that $\mathrm{Hom}_{\mathrm{D}^b_H(\mathfrak{X})}(K, L[1])$ vanishes. Let $f : \mathfrak{X} \to \operatorname{Spec}\mathbb{C}$ be the structural morphism. As we have

$$\mathrm{Hom}_{\mathrm{D}^b_H(\mathfrak{X})}(K, L[1]) \simeq \mathrm{Hom}_{\mathrm{D}_H(\operatorname{Spec}\mathbb{C})}(\mathbb{Q}_{\operatorname{Spec}\mathbb{C}}, f_* \mathscr{H}\!om(K, L)[1]),$$

by Corollary 3.20 and the usual orthogonality in the derived category of mixed Hodge structures, it suffices to show that $\mathscr{H}\!om(K, L)$ has weights $\geqslant 0$, which results from Proposition 3.18.

Now objects of weight 0 generate the ∞-category under finite colimits. Indeed by dévissage it suffice to show that objects of the perverse heart which are pure are in the ∞-category generated by the complexes of weight 0, but now this is only a shift. □

Remark 3.22. The above Corollary 3.21 is *false* for general algebraic stacks, as shown by Sun (following an example that goes back to Drinfeld) in [Sun12]. Indeed one can show that the pushforward of the constant sheaf by the quotient map $\pi\colon \operatorname{Spec}(\mathbb{C}) \to \mathrm{B}E$ with $\mathrm{B}E$ the classifying space of an elliptic curve is not semi-simple although π is smooth and proper.

3.4 Nearby cycles.

Let $\mathfrak{X}$ be an algebraic stack locally of finite type over $\mathbb{C}$ and let $f : \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ be a function. We have the following diagram:

$$\begin{array}{ccccc} \mathfrak{U} & \xrightarrow{j} & \mathfrak{X} & \xleftarrow{i} & \mathfrak{Z} \\ \downarrow{\scriptstyle f_\eta} & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_s} \\ \mathbb{G}_{m,\mathbb{C}} & \xrightarrow{j} & \mathbb{A}^1_{\mathbb{C}} & \xleftarrow{i} & \{0\} \end{array} . \tag{3.22.1}$$

We wish to construct a nearby cycle functor

$$\Psi_f \colon \mathrm{D}_H(\mathfrak{U}) \to \mathrm{D}_H(\mathfrak{Z})$$

with good properties. We begin with the unipotent part of the nearby cycle following [CHS24], whose authors give a wonderful interpretation of Ayoub's unipotent nearby cycle functor.

In [CHS24, Definition 2.22], Cass, van den Hove and Scholbach define a $\mathbb{Q}$-linear stable ∞-category Nilp (denoted by $\mathrm{Nilp}_{\mathbb{Q}}$ in *loc. cit.*) whose objects can be interpreted as pairs (K, N) with $K \in \mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}$ a complex of graded $\mathbb{Q}$-vector spaces and $N : K(1) \to K$ is a locally nilpotent map of graded complexes, where $K(1) := K \otimes_{\mathbb{Q}} \mathbb{Q}(1)$ and $\mathbb{Q}(1)$ is the graded $\mathbb{Q}$-vector space placed in degree -1. This ∞-category is compactly generated by objects $(\mathbb{Q}(k), \mathbb{Q}(k+1) \xrightarrow{0} \mathbb{Q}(k))$, and if $(K, N) \in \mathrm{Nilp}$ is compact, its underlying complex K is compact and the operator N is nilpotent. Now for any stably presentable ∞-category $\mathcal{C}$ with and action of $\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}$, on can define

$$\mathrm{Nilp}\,\mathcal{C} := \mathcal{C} \otimes_{\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}} \mathrm{Nilp},$$

which has a similar description as pairs (c, N) with $c \in \mathcal{C}$ and $N : c(1) \to c$, which is nilpotent if c is compact.

The relation of this construction with unipotent nearby cycles Υ_f comes from the desire to have a monodromy operator on Υ_f, thus a lift of Υ as a functor

$$\mathrm{D}_H(\mathfrak{U}) \to \mathrm{Nilp}\,\mathrm{D}_H(\mathfrak{Z}).$$

This is made possible thanks to the following observation.

Proposition 3.23. *Let $p : \mathbb{G}_{m\mathbb{C}} \to \mathrm{Spec}\,\mathbb{C}$ be the projection. Then the pushforward*

$$p_* \colon \mathrm{D}_{HT}(\mathbb{G}_m) \to \mathrm{D}_{HT}(\mathbb{C})$$

exhibits the ∞-category $\mathrm{D}_{HT}(\mathbb{G}_m)$ as $\mathrm{Nilp}\,\mathrm{D}_{HT}(\mathbb{C})$. Here we have denote by $\mathrm{D}_{HT}(X)$ the full subcategory of $\mathrm{D}_H(X)$ of mixed Hodge Tate modules over X, that is the ∞-category generated under colimits and shifts by the $\mathbb{Q}(i)$ for $i \in \mathbb{Z}$.

Proof. This is the combination of [CHS24, Corollary 2.20 and Lemma 2.17], that we reproduce for the reader's convenience. First we remark that the functor p_* indeed preserves Hodge Tate objects because $p_*\mathbb{Q}(n) \simeq \mathbb{Q}(n) \oplus \mathbb{Q}(n+1)[1]$. Now, it is clear that p_* preserves colimits as its left adjoint p^* preserves compact objects, and it is also conservative because if $p_*M = 0$ then for each $n \in \mathbb{Z}$ the mapping spectra $\mathrm{Map}_{\mathrm{D}_{HT}(\mathbb{G}_{m\mathbb{C}})}(\mathbb{Q}(n), M) \simeq \mathrm{Map}_{\mathrm{D}_{HT}(\mathbb{C})}(\mathbb{Q}(n), p_*M)$ vanishes, so that $M = 0$. Thus by Barr-Beck's theorem, the functor p_* upgrades to an equivalence

$$\mathrm{D}_H(\mathbb{G}_{m\mathbb{C}}) \xrightarrow{\sim} \mathrm{Mod}_{p_*\mathbb{Q}}(\mathrm{D}_H(\mathbb{C})).$$

Moreover the image under the Hodge realisation of [CHS24, Lemma 2.18] tells us that $p_*\mathbb{Q}$ is, as a commutative algebra object, the image under the symmetric monoidal functor

$$i\colon \mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}} \to \mathrm{D}_{HT}(\mathbb{C})$$

that sends $\mathbb{Q}(1)$ to $\mathbb{Q}(1)$ of the split square zero extension $\Lambda := \mathbb{Q} \otimes \mathbb{Q}(-1)[-1]$. Thus we have that

$$\mathrm{D}_{HT}(\mathbb{G}_{m\mathbb{C}}) \simeq \mathrm{Mod}_{p_*\mathbb{Q}}(\mathrm{D}_{HT}(\mathbb{C})) \simeq \mathrm{D}_{HT}(\mathbb{C}) \otimes_{\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}} \mathrm{Mod}_{\Lambda}(\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}).$$

and the equivalence $\mathrm{Mod}_{\Lambda}(\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}) \simeq \mathrm{Nilp}$ of [CHS24, Lemma 2.9] finishes the proof. □

We come back to the setup (3.22.1), and define, as [CHS24, Definition 3.1]:

Definition 3.24. The unipotent nearby cycle functor is the composite

$$\Upsilon_f\colon \mathrm{D}_H(\mathfrak{U}) \to \mathrm{Mod}_{f_\eta^* p^* p_* \mathbb{Q}}(\mathrm{D}_H(\mathfrak{U})) \simeq \mathrm{Nilp}\,\mathrm{D}_H(\mathfrak{U}) \xrightarrow{i^* j_*} \mathrm{Nilp}\,\mathrm{D}_H(\mathfrak{Z})$$

where the first functor is the way one can see any object $K \in \mathrm{D}_H(\mathfrak{U})$ as a $f_\eta^* p^* p_* \mathbb{Q}$-module with the counit $f_\eta^* p^* p_* \mathbb{Q} \otimes K \to K$, and the second is expressing that $i^* j_*$ preserves the nilpotent structure.

In fact as they show in [CHS24, Section 3.1], there is a better way to look at the above definition. Indeed the construction of the operations for stacks in Theorem 3.1 goes by extending the functor D_H on schemes to a lax monoidal functor

$$\mathrm{D}_H\colon \mathrm{Corr}_{\mathbb{C}} \to \mathrm{Pr}^L_{\mathrm{St}},$$

where $\mathrm{Corr}_{\mathbb{C}}$ is the ∞-category of of correspondences of stacks over $\mathbb{C}$: its objects are higher algebraic stacks locally of finite type over $\mathbb{C}$ and its morphisms are informally diagrams

$$Y \xleftarrow{g} Z \xrightarrow{f} X$$

of algebraic stacks locally of finite type over $\mathbb{C}$. The functor sends X to $\mathrm{D}_H(X)$, and the above morphism to

$$g_! f^*\colon \mathrm{D}_H(Y) \to \mathrm{D}_H(X).$$

In fact, the projection formulae even allow us to have a functor with values in $\mathrm{Mod}_{\mathrm{D}_H(\mathbb{C})}(\mathrm{Pr}^L_{\mathrm{St}})$.

Now, unipotent nearby cycles can be upgraded to a highly coherent natural transformation $\Upsilon\colon \mathrm{D}_{H\eta} \to \mathrm{Nilp}\,\mathrm{D}_{Hs}$ of lax monoidal functors

$$\mathrm{Corr}^{\mathrm{pr,sm}}_{\mathbb{A}^1_{\mathbb{C}}} \to \mathrm{Pr}^L_{\mathrm{gr}},$$

where $\mathrm{Cor}^{\mathrm{pr,sm}}_{\mathbb{A}^1_{\mathbb{C}}}$ is the plain subcategory of $\mathrm{Corr}_{\mathbb{A}^1_{\mathbb{C}}} = (\mathrm{Corr}_{\mathbb{C}})_{/\mathbb{A}^1_{\mathbb{C}}}$ whose morphisms are the roofs $Y \xleftarrow{g} Z \xrightarrow{f}$ over $\mathbb{A}^1$ with g proper and f smooth, and $\mathrm{Pr}^L_{\mathrm{gr}} := \mathrm{Mod}_{\mathrm{Mod}_{\mathbb{Q}}^{\mathbb{Z}}}(\mathrm{Pr}^L_{\mathrm{St}})$ is the ∞-category of presentable ∞-categories with an action of graded complexes of vector spaces.

Here, $\mathrm{D}_{H\eta}$ is the restriction of the composition

$$\mathrm{D}_{H\eta}\colon \mathrm{Corr}_{\mathbb{A}^1} \xrightarrow{-\times_{\mathbb{A}^1}\mathbb{G}_m} \mathrm{Corr}_{\mathbb{G}_m} \xrightarrow{\mathrm{D}_H} \mathrm{Mod}_{\mathrm{D}_{HT}(\mathbb{G}_{m\mathbb{C}})}(\mathrm{Pr}^L_{\mathrm{St}}) \xrightarrow{U} \mathrm{Mod}_{\mathrm{D}_{HT}(\mathbb{C})}(\mathrm{Pr}^L_{\mathrm{St}}) \to \mathrm{Pr}^L_{\mathrm{gr}}$$

with $U : \mathrm{D}_{HT}(\mathbb{G}_m) \to \mathrm{D}_{Ht}(\mathbb{C})$ the forgetful functor, and D_{Hs} is the restriction of the composition

$$\mathrm{D}_{Hs}\colon \mathrm{Corr}_{\mathbb{A}^1_{\mathbb{C}}} \xrightarrow{-\times_{\mathbb{A}^1}\{0\}} \mathrm{Corr}_{\mathbb{C}} \xrightarrow{\mathrm{D}_H} \mathrm{Pr}^L_{\mathrm{gr}} \xrightarrow{\mathrm{Nilp}} \mathrm{Pr}^L_{\mathrm{gr}}.$$

Theorem 3.2 in *loc. cit.* is:

Theorem 3.25 (Cass, van den Hove, Scholbach)**.** *There is a natural transformation of lax monoidal functors* $\mathrm{Corr}^{\mathrm{pr,sm}}_{\mathbb{A}^1_{\mathbb{C}}} \to \mathrm{Pr}^L_{\mathrm{gr}}$

$$\Upsilon \colon \mathrm{D}_{H\eta} \to \mathrm{Nilp}\, \mathrm{D}_{Hs},$$

whose evaluation at a stack $X/\mathbb{A}^1$ *is the functor defined in Definition 3.24*

Proof. The proof is *verbatim* the same as their proof in [CHS24, Section 3.3]. The only thing to check is that over stacks, where our categories are not compactly generated (but over schemes everything is compactly generated hence for example mixed Hodge Tate categories are rigid), all functors in the play indeed preserve colimits. Only j_* could be a problem, but the preservation of colimits can be checked on a presentation and it thus follows from smooth base change. Also, $\mathrm{Nilp}\, \mathrm{D}_{Hs}$ stays a h-sheaf because Nilp is compactly generated, hence the functor $\otimes_{\mathrm{Mod}^{\mathbb{Z}}_{\mathbb{Q}}} \mathrm{Nilp}$ commutes with limits. □

We will denote by $\Upsilon_f \colon \mathrm{D}_H(\mathfrak{U}) \to \mathrm{D}_H(\mathfrak{Z})$ the functor obtained after applying Υ to $f : \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ and forgetting the monodromy locally nilpotent operator N. The fact that Υ above is a lax monoidal natural transformation expresses the usual compatibilities of Υ_f with proper pushforward and smooth pullbacks, together with a natural transformation

$$\Upsilon_{f_1} \boxtimes \Upsilon_{f_2} \to \Upsilon_{f \times g}$$

compatible with the nilpotent operators when one has two functions $f_i \colon \mathfrak{X}_i \to \mathbb{A}^1_{\mathbb{C}}$.

Proposition 3.26. *Let* $f : \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ *be a function on a algebraic stack locally of finite type over* $\mathbb{C}$*. The functor* $\Upsilon_f[-1]$ *preserves* D^b_H*, is perverse* t*-exact and commutes with the external tensor product.*

Proof. The functor

$$\mathrm{rat} \colon \mathrm{D}_H \to \mathrm{D}_B$$

commutes with Υ_f because it commutes with the 6-operations. In top of this, it detects constructibility and is conservative on constructible objects. Moreover, by taking a presentation $\pi : X \to \mathfrak{X}$ of $\mathfrak{X}$, it suffices to prove the proposition for Υ_f with $f : X \to \mathbb{A}^1_{\mathbb{C}}$ a function on a scheme. Thus the proof of the proposition reduces to the case of a function of a scheme, on analytic sheaves. The comparison [CHS24, Example 3.19] of Υ in this case, with Beilinson's unipotent nearby cycles functor [Bei87a] imply the result, because for analytic sheaves the functor $\Upsilon_f[-1]$ is perverse t-exact and commutes with the external tensor product.

□

Corollary 3.27. *On* D^b_H*, there is a natural equivalence*

$$\mathbb{D}_{\mathfrak{Z}} \circ \Upsilon_f \simeq \Upsilon_f \circ \mathbb{D}_{\mathfrak{U}}(-1).$$

Proof. Using the formula [CHS24, (3.4)], we have that

$$\Upsilon_f \simeq \mathrm{colim}(\cdots \to i^* j_*(-)(-2)[-2] \to i^* j_*(-)(-1)[-1] \to i^* j_* M)$$

where the map $\mathbb{Q}(-1)[-1] \to \mathbb{Q}$ is the composition of the augmentation $p_*\mathbb{Q} \to \mathbb{Q}$ with the inclusion of $\mathbb{Q}(-1)[-1] \to p_*\mathbb{Q}$ with $p : \mathbb{G}_m \to \mathrm{Spec}\, \mathbb{C}$. Thus, there is a natural map

$$\mathbb{D}_{\mathfrak{Z}} \circ \Upsilon_f \to \Upsilon_f \circ \mathbb{D}_{\mathfrak{U}}(-1).$$

as duality commutes with $i^* j_*(-)(-n)[-n]$, and it is an equivalence because the colimit eventually stabilizes for constructible objects. □

Proposition 3.28. *Let $f : \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ be a function on a algebraic stack locally of finite type over $\mathbb{C}$. There is an exact triangle*

$$\Upsilon_f[-1] \xrightarrow{N} \Upsilon_f(-1)[-1] \to i^* j_*.$$

Proof. This is [CHS24, Proposition 3.9], shifted by -1. □

Proposition 3.29. *Let X be a reduced and separated finite type $\mathbb{C}$-scheme and let $f : X \to \mathbb{A}^1_{\mathbb{C}}$ be a function. Then there is a natural isomorphism of functors*

$$\Upsilon_f[-1] \simeq \psi_{f,1}$$

on the triangulated category $\mathrm{D}^b(\mathrm{MHM}(U))$, where $U = f^{-1}(\mathbb{G}_m)$ and $\psi_{f,1}$ is the unipotent nearby cycle defined by Saito in [Sai90b].

Proof. As both functors are t-exact it suffices to construct an isomorphism on $\mathrm{MHM}(U)$. By [Sai90a, Proposition 1.3] for $K \in \mathrm{MHM}(U)$ there is a canonical isomorphism

$$\mathrm{colim}_n\ {}^{\mathrm{p}}\mathrm{H}^{-1} i^* j_*(K \otimes f^* E_n) \xrightarrow{\sim} \psi_{f,1} K.$$

The proof of the comparison with Beilinson construction [CHS24, Example 3.19] gives in fact that

$$\Upsilon_f(K) \simeq \mathrm{colim}_n i^* j_*(K \otimes f^* E_n),$$

so that the compatibility with colimits of the t-structure gives the result. □

Now, for each $n \in \mathbb{N}^*$ we may consider π_n the elevation to the n-th power in $\mathbb{A}^1_{\mathbb{C}}$, and form the cartesian square

$$\begin{array}{ccc} \mathfrak{X}_n & \xrightarrow{e_n} & \mathfrak{X} \\ \downarrow{\scriptstyle f_n} & \lrcorner & \downarrow{\scriptstyle f} \\ \mathbb{A}^1_{\mathbb{C}} & \xrightarrow{\pi_n} & \mathbb{A}^1_{\mathbb{C}} \end{array}.$$

Then by functoriality of Υ obtain a system of functors $(\Upsilon_{f_n} \circ e_n^*)_{n \in (\mathbb{N}^*)^{\mathrm{op}}}$, where we still denote by e_n the restriction of e_n to the inverse image of $\mathbb{G}_m$ and we can set

Definition 3.30. The total nearby cycle functor is the functor

$$\Psi_f \colon \mathrm{D}_H(\mathfrak{U}) \to \mathrm{D}_H(\mathfrak{Z})$$

defined as

$$\Psi_f := \mathrm{colim}_{n \in (\mathbb{N}^*)^{\mathrm{op}}} \Upsilon_{f_n} \circ e_n^*.$$

Proposition 3.31. *Let $f : X \to \mathbb{A}^1_{\mathbb{C}}$ be a function on a reduced and separated finite type $\mathbb{C}$-scheme. Then there is a natural equivalence of triangulated functors*

$$\Psi_f[-1] \simeq \Psi_f^S$$

on $\mathrm{D}^b(\mathrm{MHM}(U))$, where Ψ_f^S is Saito's nearby cycle functor. In particular, Ψ_f preserves constructible objects.

Proof. First, because the t-structure is compatible with colimits and each e_n^* are perverse t-exact, it is clear that Ψ_f is perverse t-exact on $\mathrm{D}(\mathrm{IndMHM}(U))$. Thus by dévissage it suffices to construct an isomorphism $\Psi_f[-1] \simeq \Psi_f^S$ of exact functors on the heart $\mathrm{MHM}(U)$.

By definition ([Sai06, Proposition 5.7]) together with the comparison of Proposition 3.29, for $M \in \mathrm{MHM}(U)$ we have

$$\Psi_f^S(M) = \Upsilon_f(M)[-1] = \Upsilon_f(M \otimes f_n^*(\pi_n)_*\mathbb{Q})[-1]$$

for n divisible enough, in other terms,

$$\Psi_f^S = \mathrm{colim}_n \Upsilon_f(- \otimes f_n^*(\pi_n)_*\mathbb{Q})[-1].$$

For each $n \in \mathbb{N}^*$ because e_n is proper we have $\Upsilon_{f_n} \simeq \Upsilon_f \circ (e_n)_*$ and now the proper projection formula and proper base change give $(e_n)_* \simeq (- \otimes f_n^*(\pi_n)_*\mathbb{Q})$, finishing the proof. ☐

Theorem 3.32. *Let $f : \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ be a function on an algebraic stack locally of finite type over $\mathbb{C}$.*

1. *The functor*

$$\Psi_f[-1] \colon \mathrm{D}_H(\mathfrak{U}) \to \mathrm{D}_H(\mathfrak{Z})$$

preserves the full subcategory D^b_H, is perverse t-exact and lax monoidal.

2. *Given another function $g : \mathfrak{Y} \to \mathbb{A}^1_{\mathbb{C}}$, the natural map*

$$\Psi_f(-) \boxtimes \Psi_g(-) \to \Psi_{f\times g}(- \boxtimes -)$$

is an equivalence.

3. *On $\mathrm{D}_{H,c}(\mathfrak{U})$ the natural transformation*

$$\Psi_f \circ \mathbb{D}_{\mathfrak{U}} \to \mathbb{D}_{\mathfrak{Z}} \circ \Psi_f$$

is an equivalence.

Proof. Thanks to the same property for Υ the functor Ψ commutes with smooth pullback, hence all those properties are local for the smooth topology and can be checked on a separated finite type $\mathbb{C}$-scheme, where they follow from the results on the underlying perverse sheaves and the comparison with Saito's functor. The last assertion follows from the result on the bounded category and t-exactness. ☐

3.5 Comparison with existing constructions.

In this section, we compare our construction to the construction of Achar [Ach13] that dealt with equivariant mixed Hodge modules, as well as with Davison's [Dav24] in which they constructed pushforwards of the constant object under a morphism from a stack.

In Achar's [Ach13], the construction goes as follows: G is an affine algebraic group acting on a complex algebraic variety X, and he defines a triangulated category $\mathrm{D}_G(X)$ of G-equivariant mixed Hodge modules on X. This triangulated has a perverse t-structure, whose heart is indeed the abelian category of G-equivariant mixed Hodge modules. The definition is the following: for a special Grothendieck topology called the *acyclic topology* in which covering are smooth G-equivariant morphism satisfying some cohomological vanishing (it is asked that for some n and universally, f satisfies that $\tau^{\leqslant n} f_* f^*$ is the identity when restricted to the abelian category of mixed Hodge modules), it turns out that the presheaf of triangulated categories $U \mapsto \mathrm{hoD}^b(\mathrm{MHM}(X))$ is a sheaf, and as X admits a covering in the acyclic topology by a U on which G acts freely (so that $[U/G]$ is a scheme), this sheaf canonically extends to G-equivariant varieties.

Proposition 3.33. *Let X be a G-variety. Then the homotopy category of $\mathrm{D}^b_H([X/G])$ coincides with Achar's category $\mathrm{D}^b_G(X)$.*

Proof. First, if G acts freely on X (for the definition of a free action, see [Ach13, Definition 6.4]), then $[X/G]$ is a scheme and the result is trivial. Then exactly for any n-acyclic map $U \to X$ such that the action of G on U is free and $a \leqslant b$ integers such that $b - a < n$, the same proof as [Ach13, Lemma 8.1] implies that the functor

$$\mathrm{D}^{[a,b]}_H([X/G]) \to \mathrm{D}^{[a,b]}_H([U/G])$$

is fully faithful. Thus the functor

$$\mathrm{hoD}^b_H([X/G]) \to \mathrm{D}_G(X)$$

is fully faithful. It is not hard to check that the heart of $\mathrm{D}^b_H([X/G])$ is also the abelian category of G-equivariant mixed Hodge modules (because they satisfy descent), so that this finishes the proof. □

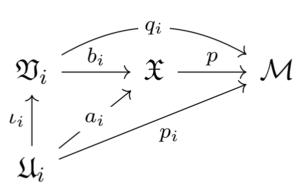
Similarly one can check that the 6 operations defined by Achar coincide with ours.

We now compare the pushforward we constructed with the one considered in [Dav24]. Let X be a smooth algebraic variety over $\mathbb{C}$, and let G be an affine algebraic groups acting on it. We are interested to the perverse cohomology groups of the object $p_!\mathbb{Q}$, where

$$p\colon \mathfrak{X} := [X/G] \to \mathcal{M}$$

is a morphism of stacks, with $\mathcal{M}$ an algebraic variety. Here is how Davison and Meinhardt proceed: the construction is very similar to the construction of the compactly supported motive of a classifying space by Totaro in [Tot16], and our proof of the comparison is inspired from the proof of [HPL21, Proposition A.7] by Hoskins and Pépin-Lehalleur and goes back to Borel. We will denote by X/G the quotient stacks instead of $[X/G]$.

First, they choose an increasing family $V_1 \subset V_2 \subset \cdots \subset V_i \subset \cdots$ of representations of G, and a subsystem $U_1 \subset U_2 \subset \cdots \subset U_i \subset \cdots$ of representations on which G acts freely (this can be done by choosing a closed embedding $G \subset \mathrm{GL}_r(\mathbb{C})$ and then setting $V_i = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^i, \mathbb{C}^r)$, and U_i is the subset of surjective linear applications). Then U_i/G is an algebraic variety, and in the case they deal with (they ask for the group G to be *special*), the quotient stack $(X \times U_i)/G$ is also a scheme by a proposition of Edidin and Graham. We denote by $\mathfrak{V}_i := (V_i \times X)/G$, $\mathfrak{U}_i := (U_i \times X)/G$. We have a commutative diagram:



Moreover the maps a_i, b_i and ι_i are smooth. Davison and Meinhardt prove that for a given $n \in \mathbb{Z}$, the object

$$^{\mathrm{p}}\mathrm{H}^n((p_i)_!\mathbb{Q}_{\mathfrak{U}_i}(-ir)[-2ir])$$

is independent of the choices for i and M large depending on n, and they denote by $^{\mathrm{p}}\mathrm{H}^n(p_!\mathbb{Q}_{\mathfrak{X}})$ the common value. We will show that indeed the canonical map

$$^{\mathrm{p}}\mathrm{H}^n((p_i)_!\mathbb{Q}_{\mathfrak{U}_i}\{-ir\}) \to {}^{\mathrm{p}}\mathrm{H}^n(p_!\mathbb{Q}_{\mathfrak{X}})$$

is an isomorphism in $\mathrm{MHM}(\mathcal{M})$, where we have denoted by $(-)\{n\} := (-)(n)[2n]$ for $n \in \mathbb{Z}$. In fact we will show better, as the result (which is classical, and see [KR24] for a vast generalization) holds universally in motives:

Proposition 3.34. *The natural map*

$$\mathrm{colim}_i (p_i)_! \mathbb{Q}_{\mathfrak{U}_i}\{-ir\} \to p_! \mathbb{Q}_{\mathfrak{X}}$$

is an equivalence in $\mathrm{DM}_{\text{ét}}(\mathcal{M}, \mathbb{Q})$*, where* $\mathrm{DM}_{\text{ét}}(\mathcal{M}, \mathbb{Q})$ *is the* ∞*-category of étale motives with rational coefficients.*

Proof. First note that the counit map

$$(b_i)_!(b_i)^! \mathbb{Q}_{\mathfrak{X}} \to \mathbb{Q}_{\mathfrak{X}}$$

is an equivalence by $\mathbb{A}^1$-invariance, because b_i is a vector bundle on $\mathfrak{X}$. As b_i is smooth of relative dimension ir, purity gives that the canonical map

$$(q_i)_! \mathbb{Q}_{\mathfrak{V}_i}\{-ir\} \to p_! \mathbb{Q}_{\mathfrak{X}}$$

is an equivalence for each $i \in \mathbb{N}$. Thus it suffices to prove that the map

$$\mathrm{colim}_i (a_i)_! \mathbb{Q}_{\mathfrak{U}_i}\{-ir\} \to \mathrm{colim}_i (b_i)_! \mathbb{Q}_{\mathfrak{V}_i}\{-ir\}$$

on $\mathfrak{X}$ induced by the counits $(\iota_i)_! \iota_i^* \to \mathrm{Id}$ is an equivalence, because applying $p_!$ would produce the seeked isomorphism.

Denote by $\pi : X \to \mathfrak{X}$ the projection. Recall that π^* is conservative, proper base change ensures that it suffices to deal with the analogous situation over X

$$\begin{array}{ccc} V_i & \xrightarrow{\beta_i} & X \\ {\scriptstyle j_i}\uparrow & \nearrow_{\alpha_i} & \\ U_i & & \end{array}$$

where every thing has been pulled back to X, and the map is now

$$\mathrm{colim}_i (\alpha_i)_! \mathbb{Q}_{U_i}\{-ir\} \to \mathrm{colim}_i (\beta_i)_! \mathbb{Q}_{V_i}\{-ir\}.$$

Everything is now a scheme. The family of pullbacks $(x^*)_{x \in X}$ is conservative on $\mathrm{DM}_{\text{ét}}(X, \mathbb{Q})$ by localisation, thus by proper base change we reduce further to the case where $X = \mathrm{Spec}\, k$ is the spectrum of a field, and each V_i is a smooth k-scheme (in fact, a vector space). Furthermore, the codimension of $W_i := V_i \setminus U_i$ is V_i goes to ∞ when $i \to \infty$, and the smoothness of α_i and β_i ensure that

$$(\alpha_i)_! \alpha_i^* \mathbb{Q}\{-ir\} \simeq (\alpha_i)_\sharp \alpha_i^* \mathbb{Q} \simeq M(U_i) \in \mathrm{DM}_{\text{ét}}(k, \mathbb{Q}),$$

and the same for $(\beta_i)_! \mathbb{Q}\{-ir\} = M(V_i)$, where for Y a smooth k-scheme, the object $M(Y)$ is the motive of Y. We are looking at

$$\mathrm{colim}_i M(U_i) \to \mathrm{colim}_i M(V_i)$$

in $\mathrm{DM}_{\text{ét}}(k, \mathbb{Q})$, with $\mathrm{codim}_{V_i}(V_i \setminus U_i) \underset{i \to \infty}{\longrightarrow} \infty$. By [HPL21, Proposition 2.13] this is an equivalence. □

Corollary 3.35. *In* $\mathrm{D}_H(\mathcal{M}) = \mathrm{Ind}\mathrm{D}^b(\mathrm{MHM}(\mathcal{M}))$ *the natural map*

$$\mathrm{colim}_i (p_i)_! \mathbb{Q}_{\mathfrak{U}_i}\{-ir\} \to p_! \mathbb{Q}_{\mathfrak{X}}$$

is an equivalence. In particular, for each $n \in \mathbb{N}$*, the natural maps*

$$^{\mathrm{p}}\mathrm{H}^n((p_i)_! \mathbb{Q}_{\mathfrak{U}_i})\{-ir\} \to \mathrm{H}^p(p_! \mathbb{Q}_{\mathfrak{X}})$$

are equivalences for i *big enough.*

Proof. The Hodge realisation

$$\rho_H \colon \mathrm{DM}_{\text{ét}} \to \mathrm{D}_H$$

extends naturally to stacks, in a way that commutes with the operations and colimits, thus the first statement is just the Hodge realisation applied to Proposition 3.34. The second statement follows from the fact that the t-structure on $\mathrm{D}_H(\mathcal{M})$ is compatible with filtered colimits, and that each $^{\mathrm{p}}\mathrm{H}^n(p_! \mathbb{Q}_{\mathfrak{X}})$ is constructible thus Noetherian. □

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