Nori motives (and mixed Hodge modules) with integral coefficients

Raphaël Ruimy, Swann Tubach

Abstract

We construct abelian categories of integral Nori motivic sheaves over a scheme of characteristic zero. The first step is to study the presentable derived category of Nori motives over a field. Next we construct an algebra in étale motives such that modules over it afford a t-structure that restricts to constructible objects. This category of integral Nori motives has the six operations and arc-descent. We finish by providing analogous constructions and results for mixed Hodge modules on schemes over the reals.

Contents

Introduction

A cohomology theory of algebraic varieties can be represented by a motivic ring spectrum, or motivic algebra, and vice versa. Finding the universal cohomology of algebraic varieties is therefore the same as finding a universal algebra in the category of étale motives. The most natural candidate is the unit object, which represents étale motivic cohomology. In characteristic zero, all known cohomology theories compare to the Betti motivic algebra, that is the algebra corresponding to the cohomology theory given by singular cohomology of the complex points. This algebra has the convenient property that modules over it afford a t-structure. In this paper, we define the Nori algebra which is universal among motivic algebras that compare to the Betti algebra and for which modules over them have a reasonable t-structure.

The above description is derived in nature: the category of étale motives is constructed out of a derived category, and we what we call a cohomology theory is a functor

$$\mathrm{R}\Gamma\colon \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{D}(\mathcal{A})$$

sending a smooth variety over a field k to a complex of objects (in some abelian category $\mathcal{A}$); this contains more information than all the H^i of this complex. There is however another construction of a universal cohomology theory, that has a more abelian nature. This construction is due to Nori. We recall it before going into more details about our construction.

Motives over a field. We fix for now a subfield k of $\mathbb{C}$. Nori's fundamental insight is that as every known cohomology theory of k-varieties can be compared to singular cohomology, one should start from the abelian groups

$$\mathrm{H}^i_{\mathrm{B}}(X, Z) := \mathrm{H}^i_{\mathrm{sing}}(X^{\mathrm{an}}, Z^{\mathrm{an}})$$

given by the singular cohomology of the analytification of any algebraic variety X over k, relative to the analytification of a closed subvariety $Z \subseteq X$. The category $\mathcal{M}(k, \mathbb{Z})$ of Nori motives over k is then an abelian category containing an object $\mathrm{H}^i(X, Z)$ which, in a sense that will be made precise, is the abelian group $\mathrm{H}^i_{\mathrm{B}}(X, Z)$ endowed with the finest structure that is preserved by algebraic maps; this mysterious structure is in fact the action of an affine group scheme over $\mathbb{Z}$: the motivic Galois group $\mathrm{G}_{\mathrm{mot}}(k)$. There is a quiver (or oriented graph) Pairs_k, constructed out of the morphisms between the $\mathrm{H}^i(X, Z)$ giving functoriality in (X, Y), boundaries for the long exact sequence of cohomology and expressing the fact that the Lefschetz object $\mathbb{Z}(-1) := \mathrm{H}^2(\mathbb{P}^1_k, \emptyset)$ is invertible (see [HMS17, Definition 9.3.2]). For Nori, a cohomology theory is a representation

$$\mathrm{H}^*\colon \mathrm{Pairs}_k \to \mathcal{A}$$

of the quiver of pairs, so that our $\mathrm{H}^i_{\mathrm{B}}$ gives a cohomology theory $\mathrm{H}^*_{\mathrm{B}}$. The abelian category of Nori motives is the universal representation of the quiver Pairs_k that factors the Betti representation in a faithful way: for every diagram

$$\begin{tikzcd}
\mathrm{Pairs}_k \arrow[r, "\mathrm{H}^*_{\mathcal{A}}"] \arrow[rr, bend left, "\mathrm{H}^*_{\mathrm{B}}"] \arrow[dr, "\mathrm{H}^*_{\mathrm{univ}}"'] & \mathcal{A} \arrow[r, "\mathrm{f}_{\mathcal{A}}"] & \mathrm{Ab}^{\mathrm{ft}} \\
& \mathcal{M}(k,\mathbb{Z}) \arrow[u, dashed, "R_{\mathcal{A}}"] \arrow[ur, "R_{\mathrm{B}}"'] &
\end{tikzcd}$$

of full arrows, with $\mathrm{f}_{\mathcal{A}}$ faithful and exact, there exists an essentially unique faithful exact functor $R_{\mathcal{A}}$ fitting in the dashed arrow. In such a diagram, one can see $\mathcal{A}$ as additional structure on singular cohomology of pairs, so that it is clear that $\mathcal{M}(k,\mathbb{Z})$ provides the best possible structure.

This abelian approach is linked to the above derived approach. The ∞-category of geometric étale motives $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Z})$ with integral coefficients is made in such way that any enriched mixed Weil cohomology theory in the sense of Cisinski and Déglise's [CD12] $\mathrm{R}\Gamma\colon \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{D}(\mathcal{A})$ factors through the canonical functor $h\colon \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Z})$ (which sends an smooth variety to its cohomological motive). The following result provides the link between Voevodsky and Nori motives.

Theorem (Nori, Harrer, Choudhury, Gallauer [Har16, CG17])**.** *There exists a Nori realisation functor*

$$\rho_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Z}) \to \mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$$

such that the composition with the Betti realisation of Nori motives $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z})) \xrightarrow{R_{\mathrm{B}}} \mathrm{D}^b(\mathrm{Ab}^{\mathrm{fr}})$ *gives the derived Betti cohomology theory* $C^*_{\mathrm{sing}}\colon \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{D}^b(\mathrm{Ab}^{\mathrm{ft}})$.

This shows that any cohomology theory *à la* Nori induces a derived cohomology theory. Conversely there exists a representation $\mathrm{Pairs}_k \to \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Z})$, therefore any derived cohomology theory induces a representation of the quiver of pairs. Giving a formal meaning to these considerations led to a new universal property of Nori motives ([HMS17, Corollary 10.1.7]): for every diagram

$$\begin{tikzcd}
\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Z}) \arrow[r, "\mathrm{H}^*_{\mathcal{A}}"] \arrow[rr, bend left, "\mathrm{H}^0\circ\rho_{\mathrm{B}}"] \arrow[dr, "\mathrm{H}^0\circ\rho_{\mathrm{N}}"'] & \mathcal{A} \arrow[r, "\mathrm{f}_{\mathcal{A}}"] & \mathrm{Ab}^{\mathrm{ft}} \\
& \mathcal{M}(k,\mathbb{Z}) \arrow[u, dashed, "R_{\mathcal{A}}"] \arrow[ur, "R_{\mathrm{B}}"'] &
\end{tikzcd}$$

of full arrows with $\mathrm{H}^*_{\mathcal{A}}$ a cohomological functor and $\mathrm{f}_{\mathcal{A}}$ a faithful exact functor, there exists a unique faithful exact functor $R_{\mathcal{A}}$ filling the dashed arrow.

Rational Nori motivic sheaves. This new point of view opened a way to construct relative versions of Nori motives. Indeed, why stop at abelian groups? Sheaves of abelian groups are so nice. Using the method of Nori with a diagram of pairs, Arapura ([Ara13]) and Ivorra ([Ivo17]) constructed abelian categories of Nori motivic sheaves that are the universal recipient of a cohomology theory of relative pairs of k-varieties modelled on constructible sheaves for Arapura, and on perverse sheaves for Ivorra. Although we will not go into details, let us say that these categories have simple universal properties but lack a critical structure that one expects when dealing with sheaves: a six functors formalism. Indeed, for a theory of sheaves $\mathcal{A}(X)$ to be efficient, notably for some reductions, it is desirable to dispose of the six operations: those are adjunctions

$$f^*\colon \mathrm{D}^b(\mathcal{A}(X)) \leftrightarrows \mathrm{D}^b(\mathcal{A}(Y))\colon f_*,$$

$$f_! \colon \mathrm{D}^b(\mathcal{A}(Y)) \leftrightarrows \mathrm{D}^b(\mathcal{A}(X)) \colon f^!,$$

for $f \colon Y \to X$ a morphism of varieties, and adjunctions

$$- \otimes F \colon \mathrm{D}^b(\mathcal{A}(X)) \leftrightarrows \mathrm{D}^b(\mathcal{A}(X)) \colon \underline{\mathrm{Hom}}(F, -)$$

for F belonging to $\mathrm{D}^b(\mathcal{A}(X))$. The abelian categories constructed by Arapura and Ivorra do not have all of the six operations, because it is very hard to exhibit some morphisms of the quivers of pairs that would induce the above operations. This is where using the new universal property of Nori motives is critical: working with rational coefficients, if instead of taking a diagram of pairs, one takes the category $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X, \mathbb{Q})$ of relative étale motives and uses the perverse Betti realisation, one can construct an abelian category $\mathcal{M}_{\text{perv}}(X, \mathbb{Q})$ of perverse motivic sheaves which has the new universal property. This is what Ivorra and S. Morel do in their paper [IM24]. As promised, using the full strength of the functoriality of $\mathrm{DM}^{\text{ét}}_{\text{gm}}(X, \mathbb{Q})$ (that affords the six operations), this allows for the construction of the six operations:

Theorem (Ivorra, Morel, Terenzi [IM24, Ter24])**.** *Over quasi-projective k-varieties, the derived category $\mathrm{D}^b(\mathcal{M}_{\text{perv}}(-, \mathbb{Q}))$ of perverse motives is endowed with the six operations.*

At that point, the two sheaf theories $\mathrm{DM}^{\text{ét}}_{\text{gm}}(-, \mathbb{Q})$ and $\mathrm{D}^b(\mathcal{M}_{\text{perv}}(-, \mathbb{Q}))$ did not talk to each other because of the way they were constructed: the first one is very derived, and its universal property is highly ∞-categorical, whereas the second one is based on abelian constructions, and had poor ∞-categorical properties. In [Tub23], the second author of the present paper proved that the constructions of Ivorra-Morel and Terenzi had canonical ∞-categorical lifts, so that the universal property of the functor $\mathrm{DM}^{\text{ét}}_{\text{gm}}(-, \mathbb{Q})$ proven by Drew and Gallauer in [DG22] would provide a realisation functor

$$\rho_{\mathrm{N}} \colon \mathrm{DM}^{\text{ét}}_{\text{gm}}(-, \mathbb{Q}) \to \mathrm{D}^b(\mathcal{M}_{\text{perv}}(-, \mathbb{Q}))$$

that commutes with the six operations. This realisation functor induces an adjunction

$$\rho_{\mathrm{N}} \colon \mathrm{DM}^{\text{ét}}(-, \mathbb{Q}) \leftrightarrows \operatorname{Ind} \mathrm{D}^b(\mathcal{M}_{\text{perv}}(-, \mathbb{Q})) \colon \rho^{\mathrm{N}}_*$$

between the associated functors of presentable ∞-categories. As ρ_{N} is monoidal, its right adjoint ρ^{N}_* is lax-monoidal and thus creates a commutative algebra object $\mathscr{N}_{(-),\mathbb{Q}} := \rho^{\mathrm{N}}_* \mathbb{Q}_{(-)}$ in the presentable ∞-category of étale motives $\mathrm{DM}^{\text{ét}}(-, \mathbb{Q})$. Hence, for every variety X the functor ρ_{N} factors through a refined realisation

$$\mathrm{DM}^{\text{ét}}(X, \mathbb{Q}) \xrightarrow{-\otimes \mathscr{N}_{X,\mathbb{Q}}} \mathrm{Mod}_{\mathscr{N}_{X,\mathbb{Q}}}(\mathrm{DM}^{\text{ét}}(X, \mathbb{Q})) \xrightarrow{\widetilde{\rho_{\mathrm{N}}}} \operatorname{Ind} \mathrm{D}^b(\mathcal{M}_{\text{perv}}(X, \mathbb{Q})).$$

Theorem ([Tub23])**.** *Let $\mathrm{DN}(-, \mathbb{Q}) := \mathrm{Mod}_{\mathscr{N}_{(-),\mathbb{Q}}}(\mathrm{DM}^{\text{ét}}(-, \mathbb{Q}))$. The induced functor*

$$\widetilde{\rho_{\mathrm{N}}} \colon \mathrm{DN}(-, \mathbb{Q}) \to \operatorname{Ind} \mathrm{D}^b(\mathcal{M}_{\text{perv}}(-, \mathbb{Q}))$$

is an equivalence. In particular, for every k-variety of structural map $p_X \colon X \to \operatorname{Spec}(k)$ the canonical map $p_X^* \mathscr{N}_{k,\mathbb{Q}} \to \mathscr{N}_{X,\mathbb{Q}}$ is an equivalence.

Furthermore, if K is a field extension of k included in $\mathbb{C}$, the map $(\mathscr{N}_{k,\mathbb{Q}})|_K \to \mathscr{N}_{K,\mathbb{Q}}$ is an equivalence.

The whole sheaf theory $\operatorname{Ind}\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(-,\mathbb{Q}))$ can therefore be recovered from the algebra $\mathscr{N}_{k,\mathbb{Q}} \in \mathrm{CAlg}(\mathrm{DM}^{\text{ét}}(k,\mathbb{Q}))$. The last part of the theorem allows to extend the functor $\mathrm{DN}(-,\mathbb{Q})$ to any $\mathbb{Q}$-scheme $p_X\colon X \to \mathrm{Spec}(\mathbb{Q})$ by setting $\mathscr{N}_{X,\mathbb{Q}} = p_X^*\mathscr{N}_{\mathrm{Spec}(\mathbb{Q}),\mathbb{Q}}$. Hence, everything can be recovered from the algebra $\mathscr{N}_{\mathrm{Spec}(\mathbb{Q}),\mathbb{Q}}$ and the functor $\mathrm{DN}(-,\mathbb{Q})$ can be enhanced into a six functors formalism thanks to the results of [AGV22].

The Nori algebra. The construction of Ivorra and Morel could only work with rational coefficients: their construction notably relies on the fact that the Verdier duality functor is perverse t-exact and can be expressed as a derived functor on the category of perverse sheaves using [Bei87]. This is not at all the case with integral coefficients as this was pointed out in [BBD82, Section 3.3]. They also use in a crucial manner the motivic unipotent nearby cycle functor constructed by Ayoub, and this functor is behaves badly with integral coefficients [Ayo07, Remarque 3.4.14]. However, the following idea to extend their work is very natural and is the starting point of the present paper: note first that in any $\mathbb{Z}$-linear stable presentable ∞-category $\mathcal{C}$ and any object A in $\mathcal{C}$, there is a Hasse fracture square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim_n A \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \arrow[d] \\ A \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r] & (\lim_n A \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \end{tikzcd}$$

where the limits are taken over the poset $\mathbb{N}^*$ of nonzero integers ordered by divisibility (so that if $A = \mathbb{Z} \in \mathrm{D}(\mathbb{Z})$, the top right object is just the profinite completion $\hat{\mathbb{Z}}$ of the integers). If one manages to construct a map

$$\mathscr{N}_{k,\mathbb{Q}} \to (\lim_n \mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Q})$, then one could use a Hasse fracture square to *define* an algebra $\mathscr{N}_k := \mathscr{N}_{k,\mathbb{Z}}$ in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})$ such that for every k-variety $p_X\colon X \to \mathrm{Spec}(k)$ the ∞-category $\mathrm{DN}(X,\mathbb{Z})$ of modules over $p_X^*\mathscr{N}_k$ in $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$ is entitled the name of integral Nori motives over X.

To do so, one solution is to study integral Nori motives over a field. As explained above, we have an abelian category $\mathcal{M}(k,\mathbb{Z})$ of Nori motives over k. Thus there are two naive possibilities for a presentable ∞-category of Nori motives over k: one could take the unbounded derived category $\mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z}))$ of the indisation of Nori motives, or take the indisation $\operatorname{Ind}\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$ of the bounded derived category of Nori motives. These two categories do not give the correct presentable category as explained in Remark 2.3.8.

Nonetheless, we first consider the ∞-category $\mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z}))$, as it is easier to get a grasp of. In fact as mentionned above, the work of Choudhury-Gallauer (Theorem 2.2.1) gives us a symmetric monoidal functor

$$\rho_{\mathrm{N}}'\colon \mathrm{DM}^{\text{ét}}(k,\mathbb{Z}) \to \mathrm{D}(\operatorname{Ind}(\mathcal{M}(k,\mathbb{Z}))).$$

Thus we may take a right adjoint F of this functor, and *define* $\mathscr{N}_k$ as the value of F at the unit object of $\mathrm{D}(\operatorname{Ind}(\mathcal{M}(k,\mathbb{Z})))$. Using rigidity for Nori motives over a field (as proven by Nori, Theorem 2.1.1), and the fact that $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$ is a full subcategory if $\mathrm{D}(\operatorname{Ind}(\mathcal{M}(k,\mathbb{Z})))$, we manage to prove (Proposition 2.3.3 and Proposition 2.3.4) that the algebra $\mathscr{N}_k$ does fit in a Hasse fracture square

$$\begin{tikzcd} \mathscr{N}_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim_n \mathbb{Z}/n\mathbb{Z} \arrow[d] \\ \mathscr{N}_{k,\mathbb{Q}} \arrow[r] & (\lim_n \mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \end{tikzcd}.$$

As a consequence, we have:

Theorem (Theorem 2.3.6 and Corollary 2.3.5)**.** *Let k be a subfield of the complex numbers. Consider the factorisation of ρ'_{N} through the modules category*

$$\overline{\rho'_{\mathrm{N}}}\colon \mathrm{Mod}_{\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})) \to \mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Z})).$$

This functor is fully faithful when restricted to the thick subcategory $\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})$ generated by the $M_k(X)(i)\otimes\mathscr{N}_k$ for X smooth over k and induces an equivalence between the latter and $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$.

Moreover, if $f\colon \mathrm{Spec}(K)\to\mathrm{Spec}(k)$ is a morphism of fields, then the canonical comparisaon map

$$f^*\mathscr{N}_k \to \mathscr{N}_K$$

is an equivalence.

Integral Nori motivic sheaves. For any qcqs $\mathbb{Q}$-scheme with a structural map $p_X\colon X\to\mathrm{Spec}(\mathbb{Q})$, we now set $\mathscr{N}_X := p_X^*\mathscr{N}_{\mathrm{Spec}(\mathbb{Q})}$. If Λ is a commutative ring, we set $\mathrm{DN}(X,\Lambda) = \mathrm{Mod}_{\mathscr{N}_X\otimes_{\mathbb{Z}}\Lambda}(\mathrm{DM}^{\text{ét}}(X,\mathbb{Z}))$; we have a six functors formalism on $\mathrm{DN}(-,\Lambda)$ by Proposition 3.1.2. We have therefore produced a presentable stable ∞-category of Nori motives. Furthermore, we have a Nori realisation functor

$$\rho_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}(X,\Lambda)\to\mathrm{DN}(X,\Lambda)$$

which is simply the functor $-\otimes\mathscr{N}_X$. It is compatible with the six functors in the following sense:

Theorem (Theorem 3.3.1)**.** *Let Λ be a commutative ring. Over qcqs $\mathbb{Q}$-schemes of finite dimension, the Nori realisation $\rho_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}(-,\Lambda)\to\mathrm{DN}(-,\Lambda)$ is compatible with the operations of type f^*, f_*, $f_!$, $f^!$, $\otimes$ and $\underline{\mathrm{Hom}}(M,-)$ for M of geometric origin.*

Then next steps will aim to produce a small abelian category from $\mathrm{DN}(X,\Lambda)$. To that end, we study the subcategory $\mathrm{DN}_{\mathrm{gm}}(X,\Lambda)$ made of Nori motives of geometric origin (Definition 3.2.1). In [RT24], we proved a categorical criterion for being geometric which is *constructibility* as well as continuity and descent properties of the category $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\Lambda)$ of geometric étale motives. The same properties hold for Nori motives, more precisely:

Theorem (Theorem 3.2.2)**.** *Let Λ be a commutative ring. The functor $\mathrm{DN}_{\mathrm{gm}}(-,\Lambda)$ is finitary. Furthermore,*

1. *If X is a qcqs Λ-finite (see Definition 3.1.3) $\mathbb{Q}$-scheme, then $\mathrm{DN}_{\mathrm{gm}}(X,\Lambda)=\mathrm{DN}(X,\Lambda)^\omega$.*
2. *If the ring Λ is regular, a Nori motive M over a qcqs finite-dimensional $\mathbb{Q}$-scheme is of geometric origin if and only if it is* constructible, *that is for any open affine $U\subseteq X$, there is a finite stratification $U_i\subseteq U$ made of constructible locally closed subschemes such that each $M|_{U_i}$ is dualisable.*

In addition, the six functors from $\mathrm{DN}(-,\Lambda)$ induce a six-functors formalism on $\mathrm{DN}_{\mathrm{gm}}(-,\Lambda)$ over quasi-excellent finite-dimensional $\mathbb{Q}$-schemes (Proposition 3.3.2) and we also have a Verdier duality functor under some mild resolution of singularities assumptions (see Proposition 3.3.3).

t-structures on geometric motives. The next step is to define a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ in order to define abelian categories of Nori motivic sheaves. By analogy with constructible sheaves, two t-structures should exist: the ordinary t-structure and the perverse t-structure. The first we construct is the ordinary one. Assume that X is of finite-type over a subfield k of $\mathbb{C}$ and recall that following Saito ([Sai90, 4.6]), the second author defined in [Tub23] a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Q})(=\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Q})))$ such that the Betti realisation functor

$$\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Q}) \to \mathrm{D}_{\mathrm{cons}}(X^{\mathrm{an}},\mathbb{Q})$$

to the category of constructible analytic complexes is t-exact when the latter is endowed with its ordinary t-structure. When k is algebraically closed, this can be used to define a t-structure integrally. Indeed, we prove that a geometric motive with coefficients $\mathbb{Z}$ is exactly a geometric motive with coefficients $\mathbb{Q}$ whose Betti realisation has an underlying $\mathbb{Z}$-lattice, and those have a t-structure (Proposition 4.1.3).

In general, we use a gluing procedure due to Vaish ([Vai20]) to derive a t-structure on a scheme X from the knowledge of a t-structure on fields. This is Theorem 4.2.3 and this gives an ordinary t-structure for integral Nori motives over any finite dimensional qcqs $\mathbb{Q}$-scheme. We denote by $\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Z})$ the heart of this t-structure.

In [Nor02], Nori showed that $\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{an}},\mathbb{Q})$ is the derived category of constructible sheaves. Following his ideas and using the adaptation of his argument to the setting of rational Nori motives in [Tub23], we show that $\mathrm{DN}_{\mathrm{gm}}$ is the derived category of ordinary Nori motives.

Theorem (Theorem 4.3.2)**.** *Let X be a qcqs finite-dimensional $\mathbb{Q}$-scheme. Then, the canonical functor*

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Z})) \to \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$$

is an equivalence.

We also give two other applications of the ordinary t-structure: firstly, the Leray spectral sequence with integral coefficients is motivic Proposition 4.4.1 and secondly, we show that $\mathrm{DN}_{\mathrm{gm}}(-,\mathbb{Z})$ satisfies arc-hyperdescent over finite-dimensional qcqs $\mathbb{Q}$-schemes Theorem 4.4.2. We therefore expect it to be the case for $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(-,\mathbb{Z})$ but we do not know how to prove it.

From the ordinary t-structure, we can construct a perverse t-structure using the usual gluing method of [BBD82].

Proposition (Proposition 5.1.1)**.** *Let X be an excellent finite-dimensional $\mathbb{Q}$-scheme endowed with a dimension function. Then, there is a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ such that the ℓ-adic realisation functor*

$$R_\ell \colon \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \to \mathrm{D}_{\mathrm{cons}}(X_{\text{proét}},\mathbb{Z}_\ell)$$

is t-exact when the right-hand side is endowed with its perverse t-structure.

We let $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$ be the category of perverse Nori motives which is defined as the heart of the above t-structure. As in the case of perverse sheaves, the Verdier duality functor exchanges $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$ with another abelian category $\mathcal{M}^{+}_{\mathrm{perv}}(X,\mathbb{Z})$ (see Remark 5.1.5). We can also define a motivic intermediate extension functor that is sent to its ℓ-adic or analytic counterpart through the appropriate realisation functors (Section 5.4). This gives a motivic intersection complex $\mathrm{IC}_{X,\mathbb{Z}}$ in $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$. We prove that the classical pairings for intersection homology can be lifted to Nori motives. We expect that Wildeshaus' intersection complex of Shimura varieties in étale motives [Wil12, Wil19] realises to the complex we defined though the functor from étale motives to Nori motives. Finally, any t-exactness property of the six functors which is true after realisation carries over to Nori motives, in particular, we can prove the hard Lefschetz theorem for Nori motives.

We then study whether $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ is the derived category of perverse Nori motives. For this the classical argument of Beilinson [Bei87] fails because unipotent nearby cycles are badly behaved with integral coefficients. Under the assumption that the abelian category $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$ has *enough torsion-free objects* in the sense of Definition 1.2.4, it is easy to prove that the canonical functor

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})) \to \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$$

is an equivalence by reducing to torsion coefficients (this is Beilinson's result) and to rational coefficients. Unfortunately, we were note able to prove that there are enough torsion-free objects, but we propose a strategy (Proposition 5.2.7) that could work to achieve this goal. This is linked to a comparison with the approach of Ivorra quoted above, which has enough torsion-free objects because the generators of the category are very explicit.

The perverse t-structure on Nori motives can also be used to prove new cases of the Artin vanishing theorem for Artin motives proved by the first named author of the present paper in [Rui22]. Indeed, we prove in Section 5.3 that the functor from étale motives to Nori motives restricts to an equivalence on Artin motives, and we use that to show that if $f\colon X \to S$ is a quasi-finite affine map of characteristic zero schemes, with regular source, the functor

$$f_!\colon \mathrm{DM}^{\text{ét},sm0}_{\mathbb{Q}-c}(X,\mathbb{Z}) \to \mathrm{DM}^{\text{ét},0}(S,\mathbb{Z})$$

is t-exact (see Proposition 5.3.5). Here the left-hand side consists of $\mathbb{Q}$-constructible smooth Artin motives and the right-hand side consist of Artin motives; both sides are endowed with their perverse homotopy t-structure (these notions were defined by the first author in [Rui22, Definition 3.2.1.1]).

Integral Mixed Hodge modules. The story we told also works for mixed Hodge modules, with simpler arguments. , [Tub23] and [Tub24b] provide an ∞-categorical enhancement of mixed Hodge modules, and a description of the objects of geometric origin as modules in $\mathrm{DM}^{\text{ét}}$ over an algebra, as introduced by Drew in [Dre18]. Thus we can define a category of mixed Hodge modules with integral coefficients as the category of mixed Hodge modules with rational coefficients having an underlying lattice. We can also define these notions for schemes over the reals (by taking Galois homotopy fixed points), and prove that objects of geometric origin are also modules over an algebra in étale motives (Theorem 6.3.3). Over $\mathrm{Spec}(\mathbb{C})$, this category is the derived category of ind-mixed Hodge structure with integral coefficients (Proposition 6.1.2). We can also prove that the stable ∞-category we define is indeed the bounded derived category of ordinary mixed modules with integral coefficients (Proposition 6.4.1). Finally, we also prove in Proposition 6.4.5 that (admissible) variations of mixed Hodge structures with integral coefficients can be seen as integral mixed Hodge modules.

Conjectural picture. One of the goal of this paper was to clarify how the existence of a motivic t-structure with rational coefficients would imply the existence of a motivic t-structure with integral coefficients. In fact in [Tub23, Theorem 4.11] the second author proved that if a motivic t-structure existed over schemes for rational motives, then étale motives and the indisation of the bounded derived category of perverse Nori motives would be equivalent. Using rigidity, we show in Proposition 3.4.6 that under the same assumption, étale motives with integral coefficients are also equivalent to Nori motives, so that they also afford a t-structure. Moreover, we show that the motives killed by the Betti realisation are the only obstruction for $\mathrm{DN}(k,\mathbb{Q})$ to be the derived category of its heart (Proposition 2.2.3). This confirms a suggestion of Ayoub in [Ayo17, Remark 2.13], and implies that the following analogue of [Ayo17, Conjecture 2.8] holds for Nori motives:

Proposition (Remark 2.2.4)**.** *Let k be a subfield of $\mathbb{C}$. Let $M \in \mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Q})$ be a geometric Nori motive and let $\mathcal{F} \in \mathrm{DN}(k,\mathbb{Q})$ be killed by the Betti realisation. Then the group*

$$\mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})}(\mathcal{F}, M) = 0$$

vanishes.

Acknowledgements

We thank very warmly Frédéric Déglise and Sophie Morel for their constant support. We would also like to thank Joseph Ayoub, Robin Carlier, Denis-Charles Cisinski, Rune Haugseng, Maxime Ramzi and Luca Terenzi for answering some of our naive questions on ∞-categories, Nori motives and étale motives. Finally, we would like to thank Fabrizio Andreatta, Luca Barbieri-Viale, Federico Binda, Niels Feld, Jan Nagel, Riccardo Pengo, Juan Diego Rojas and Alberto Vezzani for their interest in this work and some useful conversations.

We were funded by the ANR HQDIAG, RR was funded by the *Prin 2022: The arithmetic of motives and L-functions* and ST had his PhD fundings given by the É.N.S de Lyon.

Notations and conventions

We freely use the language of higher categories and higher algebra of [Lur09, HA, SAG]. We will denote by Cat_∞ the ∞-category of small ∞-categories, Pr^{L} (resp. $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$) the ∞-category of presentable (resp. stable presentable) ∞-category, $\mathrm{Cat}_\infty^{\mathrm{perf}}$ the ∞-category of stable and idempotent complete small ∞-categories, and $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{St}}$ the ∞-category of compactly generated stable ∞-categories. Those four ∞-categories are endowed with the Lurie tensor product, and the indisation functor Ind is an equivalence between $\mathrm{Cat}_\infty^{\mathrm{perf}}$ and $\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{St}}$, of inverse the functor sending a compactly generated ∞-category $\mathcal{C}$ to its full subcategory $\mathcal{C}^\omega$ of compact objects.

If $\mathcal{C}$ is a symmetric monoidal ∞-category and $A \in \mathrm{CAlg}(\mathcal{C})$ is a commutative algebra object in $\mathcal{C}$, we denote by $\mathrm{Mod}_A(\mathcal{C})$ the ∞-category of A-modules in $\mathcal{C}$. If $\mathcal{C} = \mathrm{Sp}$ is the ∞-category of spectra we omit Sp from the notation: Mod_A. If $\mathcal{C}$ is a stable ∞-category, and $a, b \in \mathcal{C}$, we denote by $\mathrm{map}_{\mathcal{C}}(a, b) \in \mathrm{Sp}$ the mapping spectrum between a and b. If Λ is an ordinary commutative ring, the abelian category of Λ-modules is denoted by Λ-mod.

All schemes we consider are of characteristic zero, so that they have a map to $\mathrm{Spec}(\mathbb{Q})$. If X is a scheme, we denote by Sch_X the category of X-schemes, by $\mathrm{Sch}_X^{\mathrm{qcqs}}$ the category of quasi-compact quasi-separated (which we will write qcqs in the rest of the paper) X-schemes, by $\mathrm{Sch}_X^{\mathrm{ft}}$ the category of X-schemes of finite type and by Sm_X the category of smooth separated X-schemes.

We will use ∞-categories of étale motives: if X is a scheme and Λ is a commutative ring, we denote by $\mathrm{DM}^{\mathrm{\acute{e}t}}(X, \Lambda)$ the ∞-category of étale motives with coefficients in Λ: to construct it, one first considers $\mathbb{A}^1$-invariant étale hypersheaves on the category Sm_X with values in Mod_Λ, and then one Tate-stabilise (see [Ayo14, CD16, Rob14]). The thick subcategory generated by the Tate twisted image of the Yoneda functor

$$M_X \colon \mathrm{Sm}_X \to \mathrm{DM}^{\mathrm{\acute{e}t}}(X, \Lambda)$$

is $\mathrm{DM}^{\mathrm{\acute{e}t}}_{\mathrm{gm}}(X, \Lambda)$ the ∞-category of motives of geometric origin (that is, it is the smallest category of $\mathrm{DM}^{\mathrm{\acute{e}t}}(X, \Lambda)$ closed under finite limits and colimits and by retracts that contains the $M_X(Y)(n)$ for $Y \in \mathrm{Sm}_X$ and $n \in \mathbb{Z}$).

1 Preliminaries.

1.1 Computing right adjoints

Recall that an ∞-category $\mathcal{C}$ is called *rigid* if it is a stably symmetric monoidal ∞-category in which every object is dualisable.

Proposition 1.1.1. *Let $\mathcal{E}_c$ be a rigid stable ∞-category and denote by $\mathcal{E}$ its indisation. Let $\mathcal{M},\mathcal{N}$ be $\mathcal{E}$-modules in Pr^{L} and let $\mathcal{A}$ be an $\mathcal{E}$-algebra in Pr^{L} which is also the indisation of a rigid ∞-category. Assume given an adjunction*

$$F\colon \mathcal{M} \leftrightarrows \mathcal{N} \colon G$$

in which the right adjoint is co-continuous and the left adjoint is $\mathcal{E}$-linear. Then the right adjoint of

$$F \otimes_{\mathcal{E}} \mathrm{Id}_{\mathcal{A}} \colon \mathcal{M} \otimes_{\mathcal{E}} \mathcal{A} \to \mathcal{N} \otimes_{\mathcal{E}} \mathcal{A}$$

is given by $G \otimes_{\mathcal{E}} \mathrm{Id}_{\mathcal{A}}$.

Proof. Denote by $\mathrm{Pr}^{\mathrm{L}}(\mathcal{E})$ the ∞-category $\mathrm{Mod}_{\mathcal{E}}(\mathrm{Pr}^{\mathrm{L}})$ of modules in Pr^{L} over $\mathcal{E}$. By [HSS17, Section 4.4] this ∞-category has a natural $(\infty, 2)$-categorical enhancement $\mathbf{Pr}^{\mathrm{L}}(\mathcal{E})$ such that the map $\mathcal{E} \to \mathcal{A}$ of Ind-rigid categories induces an 2-functor

$$- \otimes_{\mathcal{E}} \mathcal{A} \colon \mathbf{Pr}^{\mathrm{L}}(\mathcal{E}) \to \mathbf{Pr}^{\mathrm{L}}(\mathcal{A}).$$

Now, the adjunction

$$f^* \colon \mathcal{M} \leftrightarrows \mathcal{N} \colon f_*$$

is a priori only an adjunction internal to Cat_∞. However, as f^* is $\mathcal{E}$-linear, the functor f_* has an $\mathcal{E}$-lax-linear structure for free (see [HA, Examples 7.3.2.8 and 7.3.2.9]). This lax structure is in fact strong by [HSS17, Proposition 4.9 (3)], and as f_* is also a left adjoint, it is a map in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{E})$, so that by [BM24, Proposition 5.2] the adjunction (f^*, f_*) is an internal adjunction in $\mathbf{Pr}^{\mathrm{L}}(\mathcal{E})$. As internal adjunctions are preserved by 2-functors (because an internal adjunction in $\mathbf{Pr}^{\mathrm{L}}(\mathcal{E})$ is just a 2-functor $\mathfrak{adj} \to \mathbf{Pr}^{\mathrm{L}}(\mathcal{E})$ from the 'walking adjunction category', see [Hau21, Section 4]), the image $(f^* \otimes_{\mathcal{E}} \mathrm{Id}_{\mathcal{A}}, f_* \otimes_{\mathcal{E}} \mathrm{Id}_{\mathcal{A}})$ of (f_*, f^*) by $- \otimes_{\mathcal{E}} \mathcal{A}$ is again an internal adjunction. Thus, the right adjoint of $f^* \otimes_{\mathcal{E}} \mathrm{Id}_{\mathcal{A}}$ is indeed $f_* \otimes_{\mathcal{E}} \mathrm{Id}_{\mathcal{A}}$.

□

1.2 Lemmas on t-structures and bounded derived categories

For t-structures we will always use cohomological conventions as in [BBD82, Section 1.3].

Lemma 1.2.1 (Lemma 3.2.18 of [RS20])**.** *Let $(\mathcal{C}_i)_i$ be a projective system of stable (resp. stable presentable) ∞-categories which admits a limit $\mathcal{C}$ in Cat_∞ (resp. $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$). Assume that for any index i, the stable ∞-category C_i is endowed with a t-structure and such that all transition functors are t-exact. Then,*

$$\mathcal{C}^{\leqslant 0} := \{M \mid \forall i, p^i(M) \in \mathcal{C}_i^{\leqslant 0}\}$$
$$\mathcal{C}^{\geqslant 0} := \{M \mid \forall i, p^i(M) \in \mathcal{C}_i^{\geqslant 0}\}$$

defines a t-structure on $\mathcal{C}$ such that each projection $p^i \colon \mathcal{C} \to \mathcal{C}_i$ is t-exact.

Lemma 1.2.2. *Let $(\mathcal{C}_i)_i$ be a filtered inductive system of stable ∞-categories which admits a colimit $\mathcal{C}$ in Cat_∞. Assume that for any index i, the stable ∞-category $\mathcal{C}_i$ is endowed with a t-structure and that all transition functors are t-exact. Then,*

$$\mathcal{C}^{\leqslant 0} := \operatorname{colim} \mathcal{C}_i^{\leqslant 0}$$

$$\mathcal{C}^{\geqslant 0} := \operatorname{colim} \mathcal{C}_i^{\geqslant 0}$$

defines a t-structure on $\mathcal{C}$ such that each $f_i \colon \mathcal{C}_i \to \mathcal{C}$ is t-exact.

Proof. Filtered colimits in Cat_∞ are computed in the category of simplicial sets. Thus, we can compute the mapping spaces as a colimit, and then as taking homotopy groups commutes with filtered colimits we obtain the orthogonality. The stability under shift is obvious, and the existence of triangles comes from the existence of triangle at some level of the colimit. □

If $\mathcal{A}$ is an abelian category, we denote by $\mathrm{D}^b(\mathcal{A})$ its bounded derived ∞-category as constructed in [BCKW19, Section 7.4].

Lemma 1.2.3. *Let $\mathcal{A} = \mathrm{colim}_i\, \mathcal{A}_i$ be a filtered colimit of small abelian categories with exact transitions. Then the canonical functor*

$$\mathrm{colim}_i\, \mathrm{D}^b(\mathcal{A}_i) \to \mathrm{D}^b(\mathcal{A})$$

is an equivalence.

Proof. Denote by

$$F\colon\ \mathrm{colim}_i\, \mathrm{D}^b(\mathcal{A}_i) \to \mathrm{D}^b(\mathcal{A})$$

the canonical functor induced by the functors $\mathrm{D}^b(\mathcal{A}_i) \to \mathrm{D}^b(\mathcal{A})$. By Lemma 1.2.2 the ∞-category $\mathrm{colim}_i\, \mathrm{D}^b(\mathcal{A}_i)$ has a t-structure, of heart $\mathrm{colim}_i\, \mathcal{A}_i = \mathcal{A}$. Thus, by the universal property of $\mathrm{D}^b(\mathcal{A})$ proved in [BCKW19, Corollary 7.4.12], there exists a canonical functor

$$G\colon \mathrm{D}^b(\mathcal{A}) \to \mathrm{colim}_i\, \mathrm{D}^b(\mathcal{A}_i)$$

determined by its restriction to $\mathcal{A}$. The restriction of $F \circ G$ to $\mathcal{A}$ is the identity of $\mathcal{A}$, thus $F \circ G \simeq \mathrm{Id}_{\mathrm{D}^b(\mathcal{A})}$. Conversely, the functor $G \circ F$ is determined by its composition with the natural functors $\iota_j\colon \mathrm{D}^b(\mathcal{A}_j) \to \mathrm{colim}_i\, \mathrm{D}^b(\mathcal{A}_i)$ which are themselves determined by their restriction to $\mathcal{A}_j$. The restriction to $\mathcal{A}_j$ of $G \circ F \circ \iota_j$ lands in

$$\mathcal{A}_j \subset \mathrm{colim}_i\, \mathcal{A}_i \subset \mathrm{colim}_i\, \mathrm{D}^b(\mathcal{A}_i)$$

and is just the canonical functor $\mathcal{A}_j \to \mathcal{A}$, thus we have that $G \circ F \simeq \mathrm{Id}_{\mathrm{colim}_i\, \mathrm{D}^b(\mathcal{A}_i)}$. □

Definition 1.2.4. Let $\mathcal{A}$ be a small abelian category and let n be an integer. An object A of $\mathcal{A}$ is

1. torsion-free if for all nonzero integers $m \in \mathbb{Z} \setminus \{0\}$ the map $A \xrightarrow{\times m} A$ is a monomorphism.
2. of n-torsion if the map $A \xrightarrow{\times n} A$ is the zero map.

We denote by $\mathcal{A}[n]$ the full subcategory of $\mathcal{A}$ that consists of n-torsion objects.

Finally, we say that $\mathcal{A}$ has *enough torsion-free objects* if for any object A in $\mathcal{A}$, there exists an epimorphism $B \twoheadrightarrow A$ with B a torsion-free object.

For A a commutative ring, we denote by Perf_A the ∞-category of perfect complexes of A-modules. It consists of dualisable objects of Mod_A.

Proposition 1.2.5. *Let $\mathcal{A}$ be a small abelian category and let n be a positive integer. Assume that $\mathcal{A}$ has enough torsion-free objects. Then the left derived functor*

$$- \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}\colon \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{A}[n])$$

of the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}\colon A \mapsto A/nA$ exists, factors through the canonical functor

$$\mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{A}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$$

and induces an fully faithful functor

$$\mathrm{D}^b(\mathcal{A}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}} \to \mathrm{D}^b(\mathcal{A}[n]).$$

Proof. The proof goes with several steps.

Step 1: The functor

$$-\otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}\colon \mathcal{A} \to \mathcal{A}[n]$$

is the left adjoint of the inclusion functor

$$\iota\colon \mathcal{A}[n] \to \mathcal{A},$$

and we can derive this adjunction to obtain an adjunction

$$-\otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}\colon \mathrm{D}^b(\mathcal{A}) \leftrightarrows \mathrm{D}^b(\mathcal{A}[n])\colon \iota.$$

Indeed, the adjunction induces an adjunction of functors between the categories of bounded complexes $\mathrm{Ch}^b(\mathcal{A})$ and $\mathrm{Ch}^b(\mathcal{A}[n])$ and we can then use Cisinski's formalism of derived functors: we endow $\mathrm{Ch}^b(\mathcal{A}[n])$ with the trivial structure of category with equivalences and fibrations (so that equivalences are quasi-isomorphisms and all maps are fibrations), and $\mathrm{Ch}^b(\mathcal{A})$ with the structure of a category with equivalences and cofibrations such that equivalences are quasi-isomorphisms and cofibrations are monomorphism with term wise torsion-free cokernels (this is indeed a structure of category with weak equivalences and cofibrations because there are enough torsion-free objects and that they are acyclic for $-\otimes_{\mathbb{Z}}\mathbb{Z}/n$). By [Cis19, Theorem 7.5.30], using that $\mathrm{D}^b(\mathcal{A}) \simeq \mathrm{Ch}^b(\mathcal{A})[W_{qiso}^{-1}]$, we obtain the existence of the adjunction on the derived categories.

Step 2: The functor $-\otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ factors trough $\mathrm{D}^b(\mathcal{A}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$. This is because of the universal property of the tensor product of idempotent complete categories: it is computed as the compact objects of the tensor product between indisation, and the latter has a universal property by definition of Lurie's tensor product of presentable ∞-categories. As we have a $\mathrm{Perf}_{\mathbb{Z}}$-linear functor $\mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{A}[n])$ and $\mathrm{D}^b(\mathcal{A}[n])$ is $\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$-linear, this step is clear.

Step 3: The functor

$$\alpha\colon \mathrm{D}^b(\mathcal{A}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}} \to \mathrm{D}^b(\mathcal{A}[n])$$

is fully faithful. Given two complexes K, L in $\mathrm{D}^b(\mathcal{A})$ and two complexes P, Q in $\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$, we have

$$\mathrm{map}_{\mathrm{D}^b(\mathcal{A})\otimes_{\mathrm{Perf}_{\mathbb{Z}}}\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}}(K \boxtimes P, L \boxtimes Q) \simeq \mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(K, L) \otimes_{\mathbb{Z}} \mathrm{map}_{\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}}(P, Q),$$

as it is proven in [HL23, Proposition 3.5.5], where $K \boxtimes P$ is the image of (K, P) under the canonical functor

$$\mathrm{D}^b(\mathcal{A}) \times \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}} \to \mathrm{D}^b(\mathcal{A}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}.$$

To prove that α is fully faithful, by dévissage it suffices to prove that it is fully faithful on objects of the form $A \boxtimes \mathbb{Z}/n\mathbb{Z}$ with A an object of $\mathcal{A}$. The image of such an object in $\mathrm{D}^b(\mathcal{A}[n])$ is $A \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. Thus, given the above formula for the mapping spectra in the tensor product, we have to show that for two objects A and B of $\mathcal{A}$, the map

$$\mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}/n \to \mathrm{map}_{\mathrm{D}^b(\mathcal{A}[n])}(A \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}, B \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})$$

is an equivalence. This is true because

$$\begin{aligned}
\mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}/n &\simeq \mathrm{cofib}(\mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, B) \xrightarrow{\times n} \mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, B)) \\
&\simeq \mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, \mathrm{cofib}(B \xrightarrow{\times n} B)) \\
&\overset{(*)}{\simeq} \mathrm{map}_{\mathrm{D}^b(\mathcal{A})}(A, \iota(B \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})) \\
&\simeq \mathrm{map}_{\mathrm{D}^b(\mathcal{A}[n])}(a \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}, b \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}),
\end{aligned}$$

where $(*)$ can be checked by taking a torsion-free resolution of b. This finishes the proof. □

Remark 1.2.6. The above functor in proposition is not essentially surjective in general, as it can be seen by taking $\mathcal{A} = \mathrm{Ab}^{\mathrm{ft}}$ the abelian category of finitely generated abelian groups, where we get the difference between $\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$ and $\mathrm{D}^b(\mathbb{Z}/n\mathbb{Z}\text{-mod}^{\mathrm{ft}})$, where $\mathbb{Z}/n\mathbb{Z}\text{-mod}^{\mathrm{ft}}$ is the abelian category of finite type $\mathbb{Z}/n\mathbb{Z}$-modules.

1.3 Change of coefficients

For Λ a commutative ring, we denote by $\mathrm{Pr}^{\mathrm{L}}_{\Lambda} := \mathrm{Mod}_{\mathrm{Mod}_{\Lambda}}(\mathrm{Pr}^{\mathrm{L}})$ the ∞-category of Λ-linear presentable ∞-categories and by $\mathrm{Cat}^{\Lambda}_{\infty} := \mathrm{Mod}_{\mathrm{Perf}_{\Lambda}}(\mathrm{Cat}^{\mathrm{perf}}_{\infty})$ the ∞-category of Λ-linear small idempotent-complete ∞-categories.

Lemma 1.3.1. *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a map in $\mathrm{CAlg}\left(\mathrm{Pr}^{\mathrm{L}}_{\mathbb{Z}}\right)$ and let G be the right adjoint to F. Assume that for any object N of $\mathcal{D}$, the natural map*

$$G(N) \otimes_{\mathbb{Z}} \mathbb{Q} \to G(N \otimes_{\mathbb{Z}} \mathbb{Q})$$

is an equivalence. If A is a commutative algebra in $\mathrm{Mod}_{\mathbb{Z}}$, denote by F_A the functor $F \otimes_{\mathrm{Mod}_{\mathbb{Z}}} \mathrm{Mod}_A$.

*Then, the functor F is fully faithful (*resp. *an equivalence) if and only if the functors $F_{\mathbb{Q}}$ and $F_{\mathbb{Z}/n\mathbb{Z}}$ are fully faithful (*resp. *an equivalence) for any nonzero integer n.*

Proof. If F is fully faithful, then [Hai22, Lemma 2.14] ensures that F_A is for any commutative algebra in $\mathrm{Mod}_{\mathbb{Z}}$. The same assertion about equivalences is obvious. This proves the "only if" part of the statement.

We now prove the converse. Assume first that the functors $F_{\mathbb{Q}}$ and $F_{\mathbb{Z}/n\mathbb{Z}}$ are fully faithful. Let M be an object of $\mathcal{C}$. We want to prove that the natural map

$$M \to GF(M)$$

is an equivalence. This can be checked after tensoring with $\mathbb{Q}$ and with $\mathbb{Z}/n\mathbb{Z}$. Since tensoring with $\mathbb{Z}/n\mathbb{Z}$ commutes with any exact functor, the map $M/n \to GF(M)/n$ being an equivalence amounts to the full faithfulness of $F_{\mathbb{Z}/n\mathbb{Z}}$.

On the other hand, our assumption on G implies that the right adjoint of $F_{\mathbb{Q}}$ sends an object of the form $N \otimes \mathbb{Q}$ to $G(N \otimes \mathbb{Q})$. Therefore, the full faithfulness of $F_{\mathbb{Q}}$ implies that the map

$$M \otimes_{\mathbb{Z}} \mathbb{Q} \to GF(M) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an equivalence. Hence, the functor F is fully faithful.

Assume now further that $F_{\mathbb{Q}}$ and $F_{\mathbb{Z}/n\mathbb{Z}}$ are essentially surjective. Let N be an object of $\mathcal{D}$. Then, we have an exact triangle

$$N \to N \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{colim}_n \; N \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z};$$

the essential surjectivity of $F_{\mathbb{Q}}$ and $F_{\mathbb{Z}/n\mathbb{Z}}$ imply that $N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ belong to the image of F. As F is fully faithful, the latter is closed under colimits and finite limits, so that N belongs to it. This finishes the proof ☐

Remark 1.3.2. An obvious case where the previous Lemma 1.3.1 applies is when $F \colon \mathcal{C} \to \mathcal{D}$ is a functor between compactly generated presentable $\mathrm{Mod}_{\mathbb{Z}}$-linear ∞-categories that preserves compact objects. Indeed, in that case the right adjoint to F commutes with colimits hence also with

$$(\mathrm{Id} \to - \otimes_{\mathbb{Z}} \mathbb{Q}) = \mathrm{colim}_n(\mathrm{Id} \xrightarrow{n} \mathrm{Id}).$$

However, we will apply this lemma in a case where $\mathcal{C}$ is *not* compactly generated.

We will also need to consider torsion objects in an ∞-category.

Definition 1.3.3. Let $\mathcal{C}$ be an object of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathbb{Z}})$ or $\mathrm{CAlg}(\mathrm{Cat}^{\mathbb{Z}}_{\infty})$ and let p be a prime number. An object M is

1. torsion when $M \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.
2. p-torsion when $M \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0$.
3. p-complete when the canonical map $M \to \lim M/p^n$ is an equivalence.

We denote by $\mathcal{C}_{\mathrm{tors}}$ (resp. $\mathcal{C}_{p-\mathrm{tors}}$, resp. $\mathcal{C}^{\wedge}_p$) the full subcategory of $\mathcal{C}$ made of its torsion (resp. p-torsion, resp. p-complete) objects.

In the case when $\mathcal{C}$ belongs to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathbb{Z}})$, the inclusion $\mathcal{C}^{\wedge}_p \subseteq \mathcal{C}$ has a left adjoint $(-)^{\wedge}_p$ given by $M \mapsto \lim M/p^n$ called the p-completion. Furthermore, the map $(-)^{\wedge}_p$ induces an equivalence

$$\mathcal{C}_{p-\mathrm{tors}} \to \mathcal{C}^{\wedge}_p$$

with an inverse given by $M \mapsto M \otimes \mathbb{Z}(p^{\infty})$ where $\mathbb{Z}(p^{\infty}) \subseteq \mathbb{Q}/\mathbb{Z}$ denotes the Prüfer p-group.

Given a map $F \colon \mathcal{C} \to \mathcal{D}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathbb{Z}})$, we can also construct its p-completion $F^{\wedge}_p \colon \mathcal{C}^{\wedge}_p \to \mathcal{D}^{\wedge}_p$ which sends $M = \lim M/p^n$ to $\lim F(M)/p^n$.

2 Derived categories of Nori motives over a field.

2.1 Reminders on Nori motives

Let k be a subfield of $\mathbb{C}$. Nori constructed a symmetric monoidal abelian category $\mathcal{M}(k, \mathbb{Z})$ that affords the following universal property (see [HMS17, Section 9 and Corollary 10.1.6]): there is a cohomological functor

$$\mathrm{H}^0_{\mathrm{N}} \colon \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k, \mathbb{Z}) \to \mathcal{M}(k, \mathbb{Z})$$

factoring the H^0 of the Betti realisation

$$\rho_{\mathrm{B}} \colon \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k, \mathbb{Z}) \to \mathrm{Mod}_{\mathbb{Z}}$$

of [Ayo10] which is universal for this property: any cohomological functor

$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k, \mathbb{Z}) \to \mathcal{C}$$

to a $\mathbb{Z}$-linear abelian category $\mathcal{C}$, such that there exists a faithful exact functor $\mathcal{C} \to \mathbb{Z}$-mod giving a factorisation of the H^0 of the Betti realisation of étale motives, will factor uniquely through $\mathrm{H}^0_{\mathrm{N}}$, in a way compatible with the various Betti realisations.

We will need a result about torsion Nori motives. As Artin motives embed fully faithfully in $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k, \mathbb{Z})$ and the Betti realisation on those is conservative and t-exact, the functor

$$\mathrm{H}^0_{\mathrm{N}} : \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k, \mathbb{Z}) \to \mathcal{M}(k, \mathbb{Z})$$

induces a faithful exact functor

$$\iota \colon \mathrm{Rep}(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}) \to \mathcal{M}(k, \mathbb{Z}) \tag{2.1.0.1}$$

where $\mathrm{Rep}(\mathrm{Gal}(\bar{k}/k), \mathbb{Z})$ is the abelian category of $\mathbb{Z}$-linear continuous representations of $\mathrm{Gal}(\bar{k}/k)$ which are finitely generated as $\mathbb{Z}$-modules.

Theorem 2.1.1 (Nori [Fak00])**.** *The functor ι defined above induces an equivalence*

$$\iota\colon \operatorname{Rep}(\operatorname{Gal}(\bar{k}/k), \mathbb{Z})_{\mathrm{tors}} \xrightarrow{\sim} \mathcal{M}(k, \mathbb{Z})_{\mathrm{tors}}$$

on the full subcategories of torsion objects.

Proof. The proof can be found in [Fak00, Theorem 6.1], but see also the proof of Proposition 5.2.5 which is a generalisation. ☐

2.2 Realisation functor to the derived category of ind Nori motives.

Let k be a subfield of $\mathbb{C}$. In this section, we recall how to construct a realisation functor

$$\rho'_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}(k, \mathbb{Z}) \to \mathrm{D}(\operatorname{Ind}\mathcal{M}(k, \mathbb{Z})),$$

compatible with the Betti realisation. Its construction goes back to Nori and Beilinson's Basic Lemma, and we will use the description given in [CG17].

By [CG17, Proposition 7.1 and its proof], there is a lax symmetric monoidal 1-functor

$$\mathrm{SmAff}_k \to \mathrm{Ch}^b(\mathcal{M}(k, \mathbb{Z})^{\mathrm{eff}})$$

from the category of affine k-schemes to the category of bounded complexes of effective Nori motives ([HMS17, Definition 9.1.3]). We can compose this functor with the functor

$$\mathrm{Ch}^b(\mathcal{M}(k, \mathbb{Z})^{\mathrm{eff}}) \to \mathrm{Ch}(\operatorname{Ind}\mathcal{M}(k, \mathbb{Z}))$$

given by the map $\mathcal{M}(k, \mathbb{Z})^{\mathrm{eff}} \to \mathcal{M}(k, \mathbb{Z})$. The construction roughly goes as follows: for each smooth affine variety X, one can construct (using Nori's Basic Lemma) a finite *cellular filtration* $\mathcal{F}$ on X by closed subschemes which computes the cohomology of X (this is similar to how the cellular filtration of a CW-complex computes its cohomology). This gives a bounded complex $C^*_{\mathcal{F}}(X)$ in $\mathrm{Ch}^b(\mathcal{M}(k, \mathbb{Z}))$ whose Betti realisation computes the singular cohomology of the complex points of X. To make a functor out of it, one has to make choices to have cellular filtrations compatible with morphisms; to that end, one considers the homotopy colimit $C^*(X)$ (in $\mathrm{Ch}(\operatorname{Ind}\mathcal{M}(k, \mathbb{Z}))$) over all possible cellular filtrations. Moreover, the composition $R_{\mathrm{B}} \circ C^*$ with the Betti realisation functor R_{B} is naturally isomorphic (as a monoidal functor) to the functor $C^*_{\mathrm{sing}}\colon X \mapsto C^*_{\mathrm{sing}}(X^{\mathrm{an}})$ sending a k-variety to the complex of singular chains over the analytic variety of its complex points. Notice finally that the functor C^* sends étale k-schemes to objects of $\mathcal{M}(k, \mathbb{Z})$ seen as complexes placed in degree 0 (where it is given by the functor ι of (2.1.0.1)).

We now use Blumberg, Gepner and Tabuda's [BGT14, Proposition 3.5]. First notice that

$$C^*\colon \mathrm{SmAff}_k \to \mathrm{Ch}(\operatorname{Ind}\mathcal{M}(k, \mathbb{Z}))$$

is a lax symmetric monoidal functor between symmetric monoidal 1-categories. Moreover, by construction of the functor C^*, we have $C^*(k) = \mathbb{Z}_k$ and for any affine k-varieties X_1 and X_2, the canonical map

$$C^*(X_1) \otimes C^*(X_2) \to C^*(X_1 \times X_2)$$

is a quasi-isomorphism, because this is true when C^* is replaced by C^*_{sing}. Setting the weak equivalences on SmAff_k to be the isomorphisms of schemes, and the class of weak equivalences W_{qiso} in $\mathrm{Ch}(\operatorname{Ind}\mathcal{M}(k, \mathbb{Z}))$ to be the class of quasi-isomorphisms, the functor C^* preserves weak equivalences.

Therefore, the proposition quoted above yields a symmetric monoidal structure on $\mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z}))$ and a symmetric monoidal functor

$$\mathrm{SmAff}_k^{\times} \to \mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z}))^{\otimes}$$

whose underlying functor is

$$\mathrm{SmAff}_k \xrightarrow{C^*} \mathrm{Ch}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z})) \xrightarrow{W_{\mathrm{qiso}}^{-1}} \mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z})).$$

Now, by [CG17, Proposition 7.1] the above functor factor through étale motives. If K/k is an extension of subfields of $\mathbb{C}$, then as any cellular filtration of a variety over k provides a cellular filtration of its base change to K, this functor is compatible with change of fields. Thus we obtained:

Theorem 2.2.1 (Choudhury-Gallauer, Harrer, Nori)**.** *Let k be a subfield of $\mathbb{C}$. There exists a symmetric monoidal functor*

$$\rho'_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}(k,\mathbb{Z}) \to \mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z}))$$

that gives back the Betti realisation when composed with the Betti realisation of Nori motives, and sends $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Z})$ to $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$. Moreover, the construction is compatible with pullback of fields.

It will be crucial for us that, rationnally, the above realisation functor is in some sense compatible with the realisation functor to $\operatorname{Ind}\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q}))$ obtained by taking the indisation

$$\rho_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}(k,\mathbb{Q}) \to \operatorname{Ind}\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q}))$$

of the restriction $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Z}) \to \mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$ of ρ'_{N}. The functor ρ_{N} is also the realisation functor constructed in [Tub23]. To that end, we first have to compare the rationalisation of $\mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z}))$ with $\operatorname{Ind}\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q}))$. We begin with a lemma.

Lemma 2.2.2. *Let k be a subfield of $\mathbb{C}$. The canonical functor*

$$\mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z})) \to \mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Q}))$$

induces an equivalence $\mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z})) \otimes \mathrm{Mod}_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Q}))$ where the tensor product is taken in Pr^{L}.

Proof. By [SAG, C.4.2.2], the functor $\mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z})) \otimes \mathrm{Mod}_{\mathbb{Q}} \to \mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Q}))$ is the stabilisation of the functor

$$\mathrm{D}^{\leqslant 0}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z})) \otimes \mathrm{Mod}_{\mathbb{Q}}^{\leqslant 0} \to \mathrm{D}^{\leqslant 0}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Q})).$$

Now, recall ([HMS17, Definition 7.1.10]) that $\mathcal{M}(k,\mathbb{Z})$ is a colimit of categories of finitely generated modules over some rings E_F, so that $\operatorname{Ind}\mathcal{M}(k,\mathbb{Z})$ is in fact the filtered colimit of $\mathrm{Mod}_{E_F}^{\heartsuit}$. Thus, the prestable category $\mathrm{D}^{\leqslant 0}(\mathcal{M}(k,\mathbb{Z}))$ is the filtered colimit of the prestable categories $\mathrm{Mod}_{E_F}^{\leqslant 0}$ as the functor $\mathrm{D}^{\leqslant 0}(-)$ is a left adjoint by [SAG, C.5.4.9]. Finally, the tensor product with $\mathrm{Mod}_{\mathbb{Q}}^{\leqslant 0}$ above is the filtered colimit of $\mathrm{Mod}_{E_F\otimes_{\mathbb{Z}}\mathbb{Q}}^{\leqslant 0}$, but this is $\mathrm{D}^{\leqslant 0}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Q}))$ by [HMS17, Lemma 7.2.2 1.]. ☐

With rational coefficients, this category $\mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Q}))$ is closely related to the ind-category $\operatorname{Ind}\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q}))$. Recall that with rational coefficients, we denote by $\mathrm{DN}(X,\mathbb{Q})$ the indization of the bounded derived category of perverse Nori motives over a scheme X.

Proposition 2.2.3. *Let k be a subfield of $\mathbb{C}$. Denote by $\mathrm{DN}(k,\mathbb{Q})^\flat$ the Verdier quotient of $\mathrm{DN}(k,\mathbb{Q})$ by the kernel of the Betti realisation. Then the presentable ∞-category $\mathrm{DN}(k,\mathbb{Q})^\flat$ admits a separated t-structure whose heart is canonically equivalent to $\operatorname{Ind}\mathcal{M}(k,\mathbb{Q})$, and the natural functor*

$$\mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Q})) \to \mathrm{DN}(k,\mathbb{Q})^\flat$$

is an equivalence.

Proof. As $\mathcal{M}(k,\mathbb{Q})$ is Noetherian, we can apply [Kra05, Proposition 3.6] to its indisation. This implies that the canonical functor

$$J\colon \operatorname{Ind}\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q})) \to \mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Q}))$$

is a Bousfield localisation (note that what is called $\mathrm{K}(\mathrm{Inj}(-))$ in *loc. cit.* is nothing but $\operatorname{Ind}\mathrm{D}^b((-)^\omega)$ by [Kra05, Proposition 2.3]).

We now only have to show that J and the Betti realisation have the same kernel. This amounts to the conservativity of the functor

$$R'_{\mathrm{B}}\colon \mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Q})) \to \mathrm{Mod}_{\mathbb{Q}}$$

obtained by deriving the Betti realisation. Take an object M to be killed by R'_{B}. As the t-structure on $\mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Q}))$ is separated, we can assume M to be in the heart. By [Hub93, Lemma 1.3] we can write $M = \operatorname{colim} M_i$ as a filtered colimit of objects M_i of $\mathcal{M}(k,\mathbb{Q})$ with injective transition maps. In particular, because $R'_{\mathrm{B}}(M)$ vanishes, so does each $R'_{\mathrm{B}}(M_i)$ and therefore M_i by conservativity of the Betti realisation on Nori motives. □

Remark 2.2.4. Using this, we can show that a conjecture of Ayoub ([Ayo17, Conjecture 2.8]) holds for Nori motives: Let k be a subfield of $\mathbb{C}$. Let $M \in \mathrm{D}^b(\mathcal{M}(k,\mathbb{Q}))$ be a complex of Nori motives and let $\mathcal{F} \in \mathrm{DN}(k,\mathbb{Q})$ be killed by the Betti realisation. Then the group

$$\mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})}(\mathcal{F}, M) = 0$$

vanishes.

Indeed, because $\mathcal{M}(k,\mathbb{Q})$ is Noetherian the above Proposition 2.2.3 and [Hub93, Proposition 2.2] imply that the functor

$$\pi\colon \mathrm{DN}(k,\mathbb{Q}) \to \mathrm{DN}(k,\mathbb{Q})^\flat$$

is fully faithful when restricted to $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q}))$. We can write $\mathcal{F} = \operatorname{colim}_i N_i$ as a filtered colimit of objects in $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q}))$, and we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})}(\mathcal{F}, M) &\simeq \lim_i \mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})}(N_i, M) \\
&\simeq \lim_i \mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})^\flat}(\pi(N_i), \pi(M)) \\
&\simeq \mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})^\flat}(\pi(\mathcal{F}), \pi(M)) \\
&\simeq \mathrm{Hom}_{\mathrm{DN}(k,\mathbb{Q})^\flat}(0, \pi(M)) = 0.
\end{aligned}$$

Proposition 2.2.3 implies that we have a canonical functor

$$\mathrm{DN}(k,\mathbb{Q}) \to \mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Q})) \simeq \mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z})) \otimes \mathrm{Mod}_{\mathbb{Q}}$$

which is fully faithful when restricted to $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q}))$: we have some control on the rational part of $\mathrm{D}(\operatorname{Ind}\mathcal{M}(k,\mathbb{Z}))$.

Corollary 2.2.5. *Let k be a subfield of $\mathbb{C}$. Then the composition*

$$\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})\otimes\mathrm{Mod}_{\mathbb{Q}}\simeq\mathrm{DM}^{\text{ét}}(k,\mathbb{Q})\xrightarrow{\rho_{\mathrm{N}}}\mathrm{Ind}\,\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q}))\to\mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Z}))\otimes\mathrm{Mod}_{\mathbb{Q}}$$

is naturally isomorphic to the functor ρ'_{N} constructed in Theorem 2.2.1 tensored by $\mathrm{Id}_{\mathrm{Mod}_{\mathbb{Q}}}$.

Proof. Indeed, the restriction of both functors to geometric motives are isomorphic: they are the realisation functor

$$\mathrm{DM}^{\text{ét}}_{\text{gm}}(k,\mathbb{Q})\to\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q})).$$

Hence, both functors coincide on compact objects and are therefore naturally equivalent. ☐

2.3 The algebra representing Nori motivic cohomology.

Let k be a subfield of $\mathbb{C}$. Thanks to Theorem 2.2.1 we have at hand a symmetric monoidal functor

$$\rho'_{\mathrm{N}}\colon\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})\to\mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Z}))$$

compatible with the Betti realisation.

Definition 2.3.1. Let F be the right adjoint of ρ'_{N}. We denote by $\mathscr{N}_k$ the $\mathbb{E}_\infty$-ring object in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})$ given by applying the lax symmetric monoidal functor F to the unit object $\mathbb{Z}_k$ of $\mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Z}))$.

Lemma 2.3.2. *Let $M\in\mathcal{M}(k,\mathbb{Z})$. Then the canonical map*

$$F(M)\otimes\mathbb{Q}\to F(M\otimes\mathbb{Q})$$

is an equivalence.

Proof. It suffices to prove that for any smooth k-scheme X and any integer $i\in\mathbb{Z}$, the map

$$\mathrm{map}_{\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})}(M_k(X)(i),F(M)\otimes\mathbb{Q})\to\mathrm{map}_{\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})}(M_k(X)(i),F(M\otimes\mathbb{Q}))$$

is an equivalence. Using that, for constructible étale motives, tensorisation by the rationals can be set out of the mapping spectra (see for example [CD16, Corollary 5.4.9]), and using the adjunction, we see that it suffices to prove that the map

$$\mathrm{map}_{\mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Z}))}(\rho'_{\mathrm{N}}(M_k(X)(i)),M)\otimes\mathbb{Q}\to\mathrm{map}_{\mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Z}))}(\rho'_{\mathrm{N}}(M_k(X)(i)),M\otimes\mathbb{Q})$$

is an equivalence. The right-hand side can be rewritten as

$$\mathrm{map}_{\mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Z}))\otimes\mathrm{Mod}_{\mathbb{Q}}}(\rho'_{\mathrm{N}}(M_k(X)(i))\otimes\mathbb{Q},M\otimes\mathbb{Q})$$

thus using Proposition 2.2.3, the above map is in fact

$$\mathrm{map}_{\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))}(\rho'_{\mathrm{N}}(M_k(X)(i)),M)\otimes\mathbb{Q}\to\mathrm{map}_{\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q}))}(\rho'_{\mathrm{N}}(M_k(X)(i))\otimes\mathbb{Q},M\otimes\mathbb{Q})$$

which is an equivalence by [BVK16, Proposition B.4.1]. ☐

Let $k \subset K$ be subfields of $\mathbb{C}$ and denote by $f\colon \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ the associated map of schemes. By construction we have a commutative square

$$\begin{tikzcd}
\mathrm{DM}^{\text{ét}}(k,\mathbb{Z}) \arrow[r, "\rho'_{\mathrm{N}}"] \arrow[d, "f^*"] & \mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Z})) \arrow[d, "f^*"] \\
\mathrm{DM}^{\text{ét}}(K,\mathbb{Z}) \arrow[r, "\rho'_{\mathrm{N}}"'] & \mathrm{D}(\mathrm{Ind}\,\mathcal{M}(K,\mathbb{Z}))
\end{tikzcd}$$

in Pr^{L}. It induces an exchange transformation between f^* and F. In particular, we have a map

$$f^*\mathscr{N}_k \to \mathscr{N}_K \tag{2.3.2.1}$$

of algebras in $\mathrm{DM}^{\text{ét}}(K,\mathbb{Z})$. We will prove that this map is an equivalence, but first we need two important propositions:

Proposition 2.3.3. *Let $\mathscr{N}_{k,\mathbb{Q}}$ be the algebra constructed in [Tub23], that is the algebra obtained by applying the right adjoint of the functor ρ_{N} to the unit object. By Corollary 2.2.5 and Lemma 2.3.2 we have a natural ring map*

$$\mathscr{N}_{k,\mathbb{Q}} \to \mathscr{N}_k \otimes_{\mathbb{Z}} \mathbb{Q}$$

in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Q})$. This map is an equivalence.

Proof. It suffices to prove that the map is an equivalence after applying the functors

$$\mathrm{map}_{\mathrm{DM}(k,\mathbb{Q})}(M_k(X)(i), -)\colon \mathrm{DM}^{\text{ét}}(k,\mathbb{Q}) \to \mathrm{D}(\mathbb{Q})$$

for X a smooth k-scheme and i an integer. Using adjunctions and Lemma 2.3.2, we see that it suffices to check that the map

$$\mathrm{map}_{\mathrm{DN}(k,\mathbb{Q})}(\rho_{\mathrm{N}}(M_k(X)(i)), \mathbb{Q}_k) \to \mathrm{map}_{\mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Q}))}(\rho'_{\mathrm{N}}(M_k(X)(i)), \mathbb{Q}_k)$$

is an equivalence, which is the case because on $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q}))$ the functor

$$\mathrm{Ind}\,\mathrm{D}^b(\mathcal{M}(k,\mathbb{Q})) \to \mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Q}))$$

is fully faithful. ☐

Recall that the chang of site from the small étale site to the smooth étale site induces a functor

$$\iota\colon \mathrm{D}(X_{\text{ét}}, R) \to \mathrm{DM}^{\text{ét}}(X, R)$$

for any commutative ring R and any scheme X. By the rigidity theorem (see [Bac21] for the most general statement), this functor is an equivalence when R is a torsion ring.

Proposition 2.3.4. *The ring map $\mathbb{1}_k \to \mathscr{N}_k$ in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})$ becomes an equivalence after passing mod n for any nonzero integer $n \in \mathbb{N}^*$: $\mathbb{1}_k/n \simeq \mathscr{N}_k/n$.*

Proof. Again to prove this, it suffices to prove that for every smooth k-scheme X and any integer $i \in \mathbb{Z}$, the map

$$\mathrm{map}_{\mathrm{DM}(k,\mathbb{Z})}(M_k(X)(i), \mathbb{1}_k \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}) \to \mathrm{map}_{\mathrm{DM}(k,\mathbb{Z})}(M_k(X)(i), \mathscr{N}_k \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})$$

is an equivalence. Using adjunctions and rigidity for étale motives, together with the fact that tensoring with $\mathbb{Z}/n\mathbb{Z}$ is a finite colimit thus goes out of the mapping spectra, it suffices to prove that the map

$$\mathrm{map}_{\mathrm{D}^b_c(k_{\text{ét}},\mathbb{Z}/n\mathbb{Z})}(\iota^{-1}(M_k(X)(i)),\mathbb{1}_k\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z})\to\mathrm{map}_{\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))}(\rho'_{\mathrm{N}}(M_k(X)(i)),\mathbb{Z}_k)\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$$

is an equivalence. The right-hand side is in fact by Proposition 1.2.5 (that we can use because $\mathcal{M}(k,\mathbb{Z})$ has enough torsionfree objects) the mapping complex in the category $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z})[n])$. By Theorem 2.1.1 the category $\mathcal{M}(k,\mathbb{Z})[n]$ is the category of finite type representations of the absolute Galois group of k with $\mathbb{Z}/n\mathbb{Z}$ coefficients. In particular, $\mathrm{D}^b(k_{\text{ét}},\mathbb{Z}/n\mathbb{Z})$ embeds in $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z})[n])$, its image being exactly complexes of representations with perfect underlying complex of $\mathbb{Z}/n\mathbb{Z}$-modules. This finishes the proof. □

Corollary 2.3.5. *Let $k\subset K$ be a map of subfield of the complex numbers. The natural map $(\mathscr{N}_k)_{|K}\to\mathscr{N}_K$ of (2.3.2.1) is an equivalence.*

Proof. Indeed, it suffices to prove this after rationalisation and passing mod n for all nonzeros integers n. The rational statement is proven in [Tub23, Corollary 4.16], and the torsion statement follows from the fact that $f^*(\mathbb{Z}/n\mathbb{Z})\simeq\mathbb{Z}/n\mathbb{Z}$ for all f, as any exact functor commutes with cofibers. □

We finish this section with the following result:

Theorem 2.3.6. *Let k be a subfield of the complex numbers. Consider the factorisation of ρ'_{N} through the modules category*

$$\overline{\rho'_{\mathrm{N}}}\colon\mathrm{Mod}_{\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(k,\mathbb{Z}))\to\mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Z})).$$

This functor is fully faithful when restricted to the thick subcategory $\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})$ of $\mathrm{DN}(k,\mathbb{Z}):=\mathrm{Mod}_{\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(k,\mathbb{Z}))$ generated by the $M_k(X)(i)\otimes\mathscr{N}_k$ for X smooth over k and induces an equivalence between the latter and $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$.

Proof. Note that $\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})$ is an idempotent complete stable ∞-category, also note that its image through $\overline{\rho'_{\mathrm{N}}}$ is contained in $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$ because the image of the $M_k(X)$ through ρ'_{N} belongs to $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$.

We first prove that $\overline{\rho'_{\mathrm{N}}}$ is fully faithful when restricted to $\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})$. To do so, it suffices to check that it is the case after being tensored over $\mathrm{Perf}_{\mathbb{Z}}$ by $\mathrm{Perf}_{\mathbb{Q}}$ and $\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$ for all nonzero integers $n\in\mathbb{N}^*$. When tensored by $\mathrm{Perf}_{\mathbb{Q}}$, using Proposition 2.3.3 and [Tub23, Proposition 4.18] we see that the functor is fully faithful. When tensored by $\mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}$, this follows from Theorem 2.1.1, Proposition 1.2.5, Proposition 2.3.4 and rigidity for étale motives. We now prove essential surjectivity. In fact, the proof of fully faithfulness rationally gives that the functor

$$\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})\to\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$$

is an equivalence when tensored by $\mathrm{Perf}_{\mathbb{Q}}$. In particular, if $K\in\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$ is a complex of Nori motives, we see that there exist an object $P\in\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})$ and an map $\overline{\rho'_{\mathrm{N}}}(P)\to M$ that becomes an equivalence after being tensored by $\mathbb{Q}$. Said in an other way, this map $\overline{\rho'_{\mathrm{N}}}(P)\to M$ has torsion cofibre, thus we may assume that M is torsion: there exists an integer $n\in\mathbb{N}^*$ such that the multiplication by n on M is zero. In this case we have that $M\oplus M[1]\simeq M\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$, so it suffices to reach $M\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$. But because M is dualisable, the image $M\otimes^L_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$ of M in $\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z})[n])$ has perfect underlying complex of $\mathbb{Z}/n\mathbb{Z}$-modules, thus lies in $\mathrm{D}^b_c(k_{\text{ét}},\mathbb{Z}/n\mathbb{Z})$. By rigidity of étale motives again, we see that $M\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$ is in the image of ρ'_{N} and therefore in that of $\overline{\rho'_{\mathrm{N}}}$, finishing the proof. □

Corollary 2.3.7. *Let k be a subfield of $\mathbb{C}$. Then the thick category $\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})$ of $\mathrm{DN}(k,\mathbb{Z})$ generated by the $M_k(X)(i)\otimes\mathscr{N}_k$ for X smooth over k is endowed with a t-structure such that the Betti realisation is t-exact.*

Remark 2.3.8. The above construction of $\mathrm{DN}(k,\mathbb{Z}) = \mathrm{Mod}_{\mathscr{N}_k}(\mathrm{DM}^{\text{ét}}(k,\mathbb{Z}))$ is a bit more involved that what the reader might have expected; there are indeed two naive possibilities for a presentable ∞-category of Nori motives over k: one could take the unbounded derived category $\mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Z}))$ of the indisation of Nori motives, or take the indization $\mathrm{Ind}\,\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$ of the bounded derived category of Nori motives. However, they do not give the correct presentable category because they cannot coincide with $\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})$.

More precisely, it is believed that the ∞-category $\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})$ affords a t-structure, that restricts to the subcategory $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Z})$ whose heart would then be $\mathcal{M}(k,\mathbb{Z})$. But then, the above naive definitions cannot give the right answer: firstly, it is known by a counterexample of Ayoub ([Ayo17, Lemma 2.4]) that the Betti realisation of the big category $\mathrm{DM}^{\text{ét}}(k,\mathbb{Q})$ is not conservative, so that it is not possible in general (at least if k is of infinite transcendence degree over $\mathbb{Q}$) that this category is of the form $\mathrm{D}(\mathrm{Ind}\,\mathcal{M}(k,\mathbb{Q}))$ because the Betti realisation of the latter *is* conservative; secondly, the category $\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})$ is not compactly generated by $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\mathbb{Z})$ when the field k has infinite cohomological dimension (take $k=\mathbb{R}$ the field of real numbers), so that $\mathrm{Ind}\,\mathrm{D}^b(\mathcal{M}(k,\mathbb{Z}))$ cannot be the right answer either. However see Proposition 2.2.3 for a result related to this discussion.

3 Relative Nori motives.

3.1 Relative Nori motives as modules in étale motives

In [Tub23] it was proven that the construction of Ivorra and Morel in [IM24] has a natural interpretation in terms of modules in étale motives. Indeed, if for X a finite type k-scheme, one denote by

$$\mathrm{DN}(X,\mathbb{Q}) := \mathrm{Ind}\,\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X))$$

the indization of the bounded derived category of perverse motives, the realisation functor

$$\rho_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}(X,\mathbb{Q}) \to \mathrm{DN}(X,\mathbb{Q})$$

factors through the ∞-category of modules in $\mathrm{DM}^{\text{ét}}(X,\mathbb{Q})$ over the algebra $\mathscr{N}_{X,\mathbb{Q}} := \pi_X^*\mathscr{N}_{k,\mathbb{Q}}$, with $\pi_X\colon X\to\mathrm{Spec}(k)$ the structural morphism, and induces an equivalence with the latter and $\mathrm{DN}(X,\mathbb{Q})$: there is a natural equivalence of functors

$$\mathrm{Mod}_{\mathscr{N}_{(-)}}(\mathrm{DM}^{\text{ét}}(-,\mathbb{Q})) \xrightarrow{\sim} \mathrm{DN}(-,\mathbb{Q}).$$

In the previous section, we constructed a commutative algebra $\mathscr{N}_k$ in $\mathrm{DM}^{\text{ét}}(k,\mathbb{Z})$ which is equivalent to $\mathscr{N}_{k,\mathbb{Q}}$ when rationalised, and to $\mathbb{1}_k/n$ when tensored with $\mathbb{Z}/n\mathbb{Z}$. For X a finite type k-scheme, we define $\mathscr{N}_X := \pi_X^*\mathscr{N}_k$ to be the pullback of $\mathscr{N}_k$ to X. Clearly, this algebra also has the property that when rationalised, it is $\mathscr{N}_{X,\mathbb{Q}}$, and that the natural map $\mathbb{1}_X/n \to \mathscr{N}_X/n$ is an equivalence for n a nonzero integer. Furthermore, if $f\colon X\to\mathrm{Spec}(\mathbb{Q})$ is the structural map, Corollary 2.3.5 implies that the natural transformation $f^*\mathscr{N}_{\mathrm{Spec}(\mathbb{Q})}\to\mathscr{N}_X$ is an equivalence. We therefore extend the definition of $\mathscr{N}_X$ to any $\mathbb{Q}$-scheme $f\colon X\to\mathrm{Spec}(\mathbb{Q})$ by setting $\mathscr{N}_X := f^*\mathscr{N}_{\mathrm{Spec}(\mathbb{Q})}$.

Definition 3.1.1. Let X be a $\mathbb{Q}$-scheme. The *presentable ∞-category of Nori motives over X* is the ∞-category

$$\mathrm{DN}(X,\mathbb{Z}) := \mathrm{Mod}_{\mathscr{N}_X}\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$$

of modules in $\mathrm{DM}^{\mathrm{\acute{e}t}}(X,\mathbb{Z})$ over the algebra $\mathscr{N}_X$.

Let Λ be a commutative ring. We then define

$$\mathrm{DN}(X,\Lambda) := \mathrm{Mod}_\Lambda \mathrm{DN}(X,\mathbb{Z}).$$

We denote by Λ_X unit object of $\mathrm{DN}(X,\Lambda)$

We can promote this construction into a functor

$$\mathrm{DN}(-,\Lambda)\colon \mathrm{Sch}_{\mathbb{Q}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\Lambda}),$$

and there is a natural transformation $\mathrm{DM}^{\mathrm{\acute{e}t}}(-,\Lambda) \to \mathrm{DN}(-,\Lambda)$: as $\mathrm{Spec}(\mathbb{Q})$ is the final object of $\mathrm{Sch}_{\mathbb{Q}}$, the functor $\mathrm{DM}^{\mathrm{\acute{e}t}}(-,\Lambda)$ actually takes values in the ∞-category of $\mathrm{DM}^{\mathrm{\acute{e}t}}(\mathrm{Spec}(\mathbb{Q}),\Lambda)$-algebras in Pr^{L}. There is a natural functor

$$-\otimes(\mathscr{N}_{\mathrm{Spec}(\mathbb{Q})}\otimes\Lambda)\colon \mathrm{DM}^{\mathrm{\acute{e}t}}(\mathrm{Spec}(\mathbb{Q}),\Lambda) \to \mathrm{Mod}_{\mathscr{N}_{\mathrm{Spec}(\mathbb{Q})}\otimes\Lambda}(\mathrm{DM}^{\mathrm{\acute{e}t}}(\mathrm{Spec}(\mathbb{Q}),\Lambda))$$

and we can define

$$\mathrm{DN}(-,\Lambda) := \mathrm{DM}^{\mathrm{\acute{e}t}}(-,\Lambda)\otimes_{\mathrm{DM}^{\mathrm{\acute{e}t}}(\mathrm{Spec}(\mathbb{Q}),\Lambda)} \mathrm{Mod}_{\mathscr{N}_{\mathrm{Spec}(\mathbb{Q})}}(\mathrm{DM}^{\mathrm{\acute{e}t}}(\mathrm{Spec}(\mathbb{Q}),\Lambda)).$$

On $\mathrm{DM}^{\mathrm{\acute{e}t}}(-,\Lambda)$, the operations f^*, $g_\sharp$ and $\otimes$ for f a morphism of schemes and g a smooth morphism of schemes are $\mathrm{DM}^{\mathrm{\acute{e}t}}(\mathrm{Spec}(\mathbb{Q}),\Lambda)$-linear. Thus, they induce analogous functors for DN: they are the $\mathscr{N}$-linearisation of the functors on $\mathrm{DM}^{\mathrm{\acute{e}t}}(-,\Lambda)$. Moreover, if g is smooth, the adjunction $(g_\sharp, g^*)$ is then an internal adjunction to the $(\infty,2)$-category of $\mathrm{DM}^{\mathrm{\acute{e}t}}(\mathrm{Spec}(\mathbb{Q}),\Lambda)$-linear presentable ∞-categories, hence the $\mathscr{N}$-linearisation of each of the two functor actually form an adjunction for DN. By [AGV22, Theorem 4.6.1 and Remark 4.6.2], we then have:

Proposition 3.1.2. *The functor*

$$\mathrm{DN}(-,\Lambda)\colon \mathrm{Sch}_{\mathbb{Q}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\Lambda})$$

is naturally part of a six functors formalism such that the functors of type f^*, $f_!$ and $\otimes$ are compatible with the forgetful functor to $\mathrm{DM}^{\mathrm{\acute{e}t}}$. We denote the tensor structure by $\otimes_{\mathrm{N}}$ and the internal Hom by $\underline{\mathrm{Hom}}_{\mathrm{N}}$.

Furthermore DN also has some continuity properties when $\mathrm{DM}^{\mathrm{\acute{e}t}}$ has.

Definition 3.1.3. Let X be a qcqs scheme and let Λ be a commutative ring. We say that X is Λ-finite if it has finite Krull dimension and

$$\sup_{x\in X, p\in\Lambda^\times} \mathrm{cd}_p(\kappa(x)) < \infty$$

where $\mathrm{cd}_p(k)$ is the Galois cohomological dimension of a field k with $\mathbb{F}_p$-coefficients.

Proposition 3.1.4. *Let $X = \lim X_i$ be a limit of qcqs étale-locally Λ-finite $\mathbb{Q}$-schemes with affine transition maps. The natural map*

$$\mathrm{colim}_i\, \mathrm{DN}(X_i,\Lambda) \to \mathrm{DN}(X,\Lambda)$$

in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is an equivalence.

Proof. The functor sending a pair $(\mathcal{C}, A)$ consisting of a symmetric monoidal presentable ∞-category $\mathcal{C}$ together with a commutative algebra A in $\mathrm{CAlg}(\mathcal{C})$ to the ∞-category $\mathrm{Mod}_A(\mathcal{C})$ of modules over A in $\mathcal{C}$ commutes with colimits: this is [AGV22, Lemma 3.5.6]. The proposition then follows from the same result for étale motives which is [EHIK21b, Lemma 5.1]. $\square$

Proposition 3.1.5. *The functor* $\mathrm{DN}(-,\Lambda)$ *has the same descent properties as* $\mathrm{DM}^{\text{ét}}(-,\Lambda)$. *In particular:*

1. *It is an étale hypersheaf and a cdh-sheaf.*
2. *It is a cdh-hypersheaf over qcqs* $\mathbb{Q}$*-schemes of finite valuative dimension.*
3. *It is an h-hypersheaf over Noetherian schemes of finite dimension.*

Proof. Note that the proposition is true if we replace DN by $\mathrm{DM}^{\text{ét}}$. Indeed, since the presentable ∞-category $\mathrm{DM}^{\text{ét}}(-,\Lambda)$ satisfies Nisnevich descent, the localisation property and proper base change, it satisfies cdh-descent as the proof of [CD19, Theorem 3.3.10] applies in this level of generality; for hyperdescent, this follows from the fact that the cdh topos is hypercomplete over qcqs schemes of finite valuative dimension by [EHIK21a, Corollary 2.4.16]. Finally, h-descent follows from [CD16, Corollary 5.5.7] as Bachmann's Rigidity Theorem [BH21, Theorem 3.1] allows to remove the hypothesis on the base scheme in *loc. cit.*

Now, let $f_\bullet \colon X_\bullet \to X$ be an hyper covering of one of the topologies considered in the proposition. By [AC23, Proposition 2.12] we have an equivalence of ∞-operads

$$\mathrm{Mod}(\mathrm{DM}^{\text{ét}}(X,\Lambda))^{\otimes} \xrightarrow{\sim} \lim_{\Delta} \mathrm{Mod}(\mathrm{DM}^{\text{ét}}(X_i,\Lambda))^{\otimes} \tag{3.1.5.1}$$

where if $\mathcal{C}$ is a symmetric monoidal ∞-category, the ∞-category $\mathrm{Mod}(\mathcal{C})$ can be informally described as pairs (A, M) with A in $\mathrm{CAlg}(\mathcal{C})$ and M in $\mathrm{Mod}_A(\mathcal{C})$. We know that the diagram

$$\mathscr{N}_X \longrightarrow (f_0)_*\mathscr{N}_{X_0} \rightrightarrows (f_1)_*\mathscr{N}_{X_1} \, \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \, \cdots$$

is a limit diagram in $\mathrm{DM}^{\text{ét}}(X,\Lambda)$. Indeed, as $\mathscr{N}$ is pulled back from X the above diagram is in fact

$$\mathscr{N}_X \longrightarrow (f_0)_*f_0^*\mathscr{N}_X \rightrightarrows (f_1)_*f_1^*\mathscr{N}_X \, \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \, \cdots,$$

so that it is a limit diagram by descent in $\mathrm{DM}^{\text{ét}}(X,\Lambda)$. Thus, the fiber $\mathrm{DN}(X,\Lambda)$ over $\mathscr{N}_X$ of the left-hand side of (3.1.5.1) corresponds to $\lim_{\Delta} \mathrm{DN}(X_i,\Lambda)$, finishing the proof. $\square$

Remark 3.1.6 (Universal property of Nori motives). Let k be a subfield of $\mathbb{C}$ and let Λ be a regular ring. By Drew and Gallauer's work [DG22], we know that $\mathrm{DM}^{\text{ét}}(-,\Lambda)$ is universal among functors

$$\mathrm{Sch}_k^{\mathrm{ft}} \to \mathrm{CAlg}(\mathrm{Pr}_\Lambda^{\mathrm{L}})$$

that satisfy $\mathbb{A}^1$-invariance, $\mathbb{P}^1$-stability, smooth base change and étale hyperdescent. We claim that $\mathrm{DN}(-,\Lambda)$ is universal among the above class of functors $\mathcal{C}$ that verify the following additional conditions: the ∞-category $\mathcal{C}_{\mathrm{liss}}(k)$ of dualisable objects in $\mathcal{C}(k)$ affords a t-structure and the induced functor $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(k,\Lambda) \to \mathcal{C}_{\mathrm{liss}}(k)$ factors the Betti realisation in a way that the functor $\mathcal{C}_{\mathrm{liss}}(k) \to \mathrm{Perf}_\Lambda$ is t-exact and conservative. Indeed, for such a $\mathcal{C}$, the universal property of Nori motives over k gives us a faithful exact functor $\mathcal{M}(k) \to \mathcal{C}_{\mathrm{liss}}(k)^{\heartsuit}$ such that the induced functor

$\mathrm{D}^b(\mathcal{M}(k)) \to \mathcal{C}_{\mathrm{liss}}(k) \to \mathrm{Perf}_\Lambda$ is the Betti realisation of Nori motives. In particular, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{DM}^{\text{ét}}(k,\Lambda) \arrow[r] \arrow[d] & \mathcal{C}(k) \\
\mathrm{DN}(k,\Lambda) \arrow[ur] &
\end{tikzcd}$$

which induces, if one denotes by $\mathscr{C}$ the commutative algebra representing cohomology with values in $\mathcal{C}(k)$, a map of algebras $\mathscr{N}_k \to \mathscr{C}$ in $\mathrm{DM}^{\text{ét}}(k,\Lambda)$. By pulling back $\mathscr{C}$ over finite type k-schemes we obtain morphisms of functors

$$\mathrm{DM}^{\text{ét}}(-,\Lambda) \to \mathrm{DN}(-,\Lambda) \to \mathrm{Mod}_{\mathscr{C}}(\mathrm{DM}^{\text{ét}}(-,\Lambda)) \to \mathcal{C},$$

thus the promised morphism of functors $\mathrm{DN} \to \mathcal{C}$.

3.2 Nori motives of geometric origin

The goal of this section is to prove the statements of [RT24] in the setting of Nori motives. Namely, we define Nori motives of geometric origin and show that they can be characterised by the property of being *constructible* which is more categorical. We also show descent properties as well as the continuity property.

Definition 3.2.1. Let X be a $\mathbb{Q}$-scheme and let Λ be a commutative ring. The category $\mathrm{DN}_{\mathrm{gm}}(X,\Lambda)$ *of Nori motives of geometric origin* (or simply *geometric Nori motives*) is the thick subcategory of $\mathrm{DN}(X,\Lambda)$ generated by the $f_\sharp(\Lambda_Y)(n)$ for $f\colon Y \to X$ smooth and n an integer.

Theorem 3.2.2. *Let Λ be a commutative ring.*

1. *The functors of type f^*, $f_!$ and $\otimes_{\mathrm{N}}$ preserve geometric objects.*
2. *Let $f\colon Y \to X$ be a morphism of qcqs $\mathbb{Q}$-schemes of finite dimension, let M and N be Nori motives over X with M geometric and let P be a Nori motive over Y. Then, the natural transformations below are equivalences:. Then, the natural transformations below are equivalences:*
   - *(a)* $\mathrm{map}_{\mathrm{DN}}(M,N)\otimes\mathbb{Q} \to \mathrm{map}_{\mathrm{DN}}(M,N\otimes\mathbb{Q})$
   - *(b)* $\underline{\mathrm{Hom}}_{\mathrm{N}}(M,N)\otimes\mathbb{Q} \to \underline{\mathrm{Hom}}_{\mathrm{N}}(M,N\otimes\mathbb{Q})$
   - *(c)* $f_*(P)\otimes\mathbb{Q} \to f_*(P\otimes\mathbb{Q})$
   - *(d)* $f^!(N)\otimes\mathbb{Q} \to f^!(N\otimes\mathbb{Q})$ *(for f of finite type)*
3. *Consider a qcqs finite-dimensional $\mathbb{Q}$-scheme X that is the limit of a projective system of qcqs finite-dimensional $\mathbb{Q}$-schemes (X_i) with affine transition maps. Then, the natural map*

   $$\mathrm{colim}\,\mathrm{DN}_{\mathrm{gm}}(X_i,\Lambda) \to \mathrm{DN}_{\mathrm{gm}}(X,\Lambda)$$

   is an equivalence.
4. *Let X be a qcqs Λ-finite $\mathbb{Q}$-scheme, then, a Nori motive M is of geometric origin if and only if it is a compact object of* $\mathrm{DN}(X,\Lambda)$.

5. *Assume that the ring Λ is ind-regular*[1]*. A Nori motive M over a qcqs finite-dimensional $\mathbb{Q}$-scheme is of geometric origin if and only if it is* constructible*, that is for any open affine $U \subseteq X$, there is a finite stratification $U_i \subseteq U$ made of constructible locally closed subschemes such that each $M|_{U_i}$ is dualisable.*

Proof. In the course of this proof, we will denote by DN_c the subcategory of constructible Nori motives.

The functors of type f^*, $f_\sharp$, $\otimes_{\mathrm{N}}$ and i_* for i a closed immersion readily preserve geometric objects. Using Nagata compactifications, the last remaining part of Assertion 1. is to prove that functors of type p_* for p proper preserve geometric objects which we will prove later.

Now, if $f\colon Y \to X$ is smooth and n is an integer, Proposition 3.1.2 implies that $f_\sharp(\Lambda_Y)(n) = \rho_{\mathrm{N}}(f_\sharp(\mathbb{1}_Y)(n))$. Now, the functor ρ_{N} being monoidal, it sends constructible motives to constructible Nori motives. Since geometric motives are constructible using [RT24, Corollary 3.7], the Nori motive $f_\sharp(\Lambda_Y)(n)$ is constructible. Hence, geometric Nori motives are constructible.

We now prove Assertion 2.(a). By adjunction, we can reduce to the case where $M = \Lambda_X$. But we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{DN}}(\Lambda_X, N) \otimes \mathbb{Q} \arrow[r] \arrow[d] & \mathrm{map}_{\mathrm{DN}}(\Lambda_X, N \otimes \mathbb{Q}) \arrow[d] \\
\mathrm{map}_{\mathrm{DM}^{\text{ét}}}(\Lambda_X, \rho_*^{\mathrm{N}} N) \otimes \mathbb{Q} \arrow[r] & \mathrm{map}_{\mathrm{DM}^{\text{ét}}}(\Lambda_X, \rho_*^{\mathrm{N}}(N \otimes \mathbb{Q}))
\end{tikzcd}$$

where the vertical arrows are equivalences by definition of the right adjoint ρ_*^{N} to the Nori realisation and the bottom horizontal arrow is an equivalence by [RT24, Proposition 3.1] together with the fact that ρ_*^{N} commutes with colimits as it is a forgetful functor.

Assertions 2.(b) and 2.(c) as well as Assertion 2.(d) for f a closed immersion then follow from 2.(a) by exactly the same proof as [RT24, Proposition 3.1]. Note that Assertion 2.(d) for a closed immersion implies that Assertion 2.(a) holds when M is only assumed to be constructible: it allows by dévissage to reduce to the case where $M = \Lambda_X$.

We now prove Assertion 4: in the Λ-finite case, the equivalence between geometric objects and compact objects can be deduced from the same result for $\mathrm{DM}^{\text{ét}}$ which is [RT24, Proposition 3.8] and [Iwa18, Lemma 2.6] which states that if a category is $\mathcal{C}$ is compactly generated, the category $\mathrm{Mod}_A(\mathcal{C})$ with A some commutative algebra is still compactly generated and that the $A \otimes M$ with M compact form a set of compact generators. This also implies Assertion 5. in the Λ-finite case: the fact that constructible objects are compact can then be proved exactly as in [RT24, Proposition 3.8].

The next step is to prove the continuity property both for geometric and for constructible Nori motives, *i.e.* that the natural maps

$$\operatorname{colim} \mathrm{DN}_{\mathrm{gm}}(X_i, \Lambda) \to \mathrm{DN}_{\mathrm{gm}}(X, \Lambda)$$

$$\operatorname{colim} \mathrm{DN}_c(X_i, \Lambda) \to \mathrm{DN}_c(X, \Lambda)$$

are equivalences (here DN_c denotes constructible Nori motives). This will in particular prove Assertion 3. As geometric motives are constructible, it suffices to prove the full faithfulness for constructible motives which can then be tested after tensoring with $\mathbb{Q}$ and with $\mathbb{Z}/p\mathbb{Z}$ for any prime number p. In the first case, it is then a consequence of Proposition 3.1.4, while in the second case, it is a consequence of Bachmann's Rigidity Theorem in the form given in [RT24, Theorem 3.2] and

---

[1] *i.e.* it is a filtered colimit of regular rings.

of Bhatt and Matthew's continuity property for torsion étale sheaves in the form stated in [RT24, Proposition 3.4] which both apply for derived rings of coefficients. The essential surjectivity of the first functor can then be proved Zariski locally so that we can assume X and the X_i to be affine in which case, the usual spreading-out properties of [Gro66, Théorème 8.10.5, Proposition 17.7.8] give the result (see the proof of [RT24, Theorem 3.5] for more details). The essential surjectivity of the second functor is then a consequence of the analogous of [RT24, Lemma 3.9] which can be proved exactly the same way:

Lemma 3.2.3. *Let M be a constructible Nori motive over a qcqs finite-dimensional $\mathbb{Q}$-scheme X. Then, there is a geometric Nori motive P as well as a map $P \to M \oplus M[1]$ whose cone C is such that $C/n \simeq C \oplus C[1]$ for some integer n.*

We can now finish the proof of Assertion 1: we have to prove that the functors of type p_* for p proper preserve geometric objects. This is a consequence of the continuity property for geometric objects and can be proved in the same fashion as in [RT24, Corollary 3.6]. Now, Assertion 1. implies Assertion 2.(d) for a general f of finite which finishes the proof of Assertion 2.

It remains to prove that constructible Nori motives are geometric. As in the case of $\mathrm{DM}^{\text{ét}}$, this can be reduced by continuity, localisation and dévissage to the case where X is the spectrum of a field and Λ is regular (see the proof of [RT24, Theorem 4.1]).

Lemma 3.2.4. *Let k be a field of characteristic zero and let Λ be a commutative ring. Then, the Artin motive functor*

$$\iota\colon \mathrm{D}(k_{\text{ét}}, \Lambda) \to \mathrm{DN}(k, \Lambda)$$

is a fully faithful monoidal functor. Furthermore any torsion Nori motive lies in its essential image.

Proof. If the result is known for $\Lambda = \mathbb{Z}$, then the result for an arbitrary ring of coefficients Λ follows from tensoring both sides with Mod_Λ: this operation preserves full faithfulness according to [Hai22, Lemma 2.14]. Now, using Lemma 1.3.1, Proposition 2.3.4 and rigidity of étale motives, it suffices to show that this result holds with $\Lambda = \mathbb{Q}$. Then, the functor ι is the indization of the Artin motive functor

$$\mathrm{D}(k_{\text{ét}}, \mathbb{Q})^\omega \to \mathrm{D}^b(\mathcal{M}(k, \mathbb{Q}))$$

which is fully faithful. Indeed we have that the canonical functor

$$\mathrm{D}^b(\mathrm{Rep}(\mathrm{Gal}(\overline{k}/k), \Lambda)) \to \mathrm{D}(k_{\text{ét}}, \mathbb{Q})^\omega$$

is an equivalence, and as all Artin motives are of weight 0 in Nori motives, the abelian category of Artin motives is semi-simple, so that the above functor is fully faithful whenever it is fully faithful on the hearts, which is proven in [HMS17, Proposition 9.4.3]. □

Let M be a constructible Nori motive. We have to show that M is geometric. Lemma 3.2.3 yields a geometric Nori motive P and a map $P \to M \oplus M[1]$ whose cone C is such that $C \otimes \mathbb{Q} = 0$.

We can then write $C = \iota D$, with D a dualisable object of $\mathrm{D}(k_{et}, \Lambda)$. In particular, it belongs to the thick stable subcategory generated by the $f_\sharp \underline{\Lambda}$ for f finite étale according to [Rui22, 2.1.2.15]. Thus, its image C through ι is constructible as ι is compatible with the functors $f_\sharp$ and f^* for f finite étale: indeed $\iota = \rho_{\mathrm{N}} \rho_!$ with $\rho_!$ the Artin motive functor to $\mathrm{DM}^{\text{ét}}$; the same result is true for $\rho_!$ using [CD16, Â§4.4.2] and for ρ_{N} by Proposition 3.1.2. Finally, $M \oplus M[1]$ is geometric and therefore M is geometric. □

Corollary 3.2.5. *Let* $\Lambda = \mathrm{colim}_i \Lambda_i$ *be a filtered colimit of rings. Then the canonical map*

$$\mathrm{colim}_i \, \mathrm{DN}_{\mathrm{gm}}(X, \Lambda_i) \to \mathrm{DN}_{\mathrm{gm}}(X, \Lambda)$$

is an equivalence.

Proof. As generators of $\mathrm{DN}_{\mathrm{gm}}(X, \Lambda)$ are clearly in the image, it suffices to prove fully faithfulness. Moreover by continuity we can assume that X is of finite type over $\mathbb{Q}$. Now this can be proven after tensorisation by $\mathbb{Q}$ and $\mathbb{Z}/p$. The rational part follows from the fact that $\mathrm{DN}_{\mathrm{gm}}(X, \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}) = \mathrm{DN}_{\mathrm{gm}}(X, \Lambda) \otimes \mathrm{Perf}_{\mathbb{Q}}$ consists of the compact objects of $\mathrm{DN}(X, \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}) = \mathrm{DN}(X, \mathbb{Q}) \otimes \mathrm{Mod}_{\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}}$ and the functor $\mathrm{DN}(X, \mathbb{Q}) \otimes \mathrm{Mod}_{\mathbb{Q} \otimes (-)}$ commutes with filtered colimits of rings. If we mod out by p, then the result follows from [RT24, Lemma 4.3]. □

Corollary 3.2.6. *Let* Λ *be an ind-regular ring. The functor* $\mathrm{DN}_{\mathrm{gm}}(-, \Lambda)$ *is an étale hypersheaf over qcqs finite-dimensional* $\mathbb{Q}$*-schemes.*

Proof. The condition of being constructible is étale local exactly as in [RT24, Lemma 2.5]. □

Corollary 3.2.7. *Let* $\Lambda \to \Lambda'$ *be a morphism between commutative rings. The canonical map*

$$\mathrm{DN}_{\mathrm{gm}}(X, \Lambda) \otimes_{\mathrm{Perf}_\Lambda} \mathrm{Perf}_{\Lambda'} \to \mathrm{DN}_{\mathrm{gm}}(X, \Lambda')$$

is an equivalence.

Proof. We claim that it is sufficient to deal with the case $\Lambda = \mathbb{Z}$. Indeed, assume that

$$\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_A \to \mathrm{DN}_{\mathrm{gm}}(X, A)$$

is an equivalence for all rings A, then we have

$$\begin{aligned}\mathrm{DN}_{\mathrm{gm}}(X, \Lambda) \otimes_{\mathrm{Perf}_\Lambda} \mathrm{Perf}_{\Lambda'} &\simeq \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_\Lambda \otimes_{\mathrm{Perf}_\Lambda} \mathrm{Perf}_{\Lambda'} \\ &\simeq \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\Lambda'} \\ &\simeq \mathrm{DN}_{\mathrm{gm}}(X, \Lambda').\end{aligned}$$

It suffices to prove fully faithfulness as generators of $\mathrm{DN}_{\mathrm{gm}}(X, \Lambda')$ clearly are in the image. Let A be a ring and let M, N be objects of $\mathrm{DN}_{\mathrm{gm}}(X, A)$. We will use that for any perfect complex of A-modules P, the canonical map

$$\mathrm{map}_{\mathrm{DN}_{\mathrm{gm}}(X,A)}(M, N) \otimes_A P \to \mathrm{map}_{\mathrm{DN}_{\mathrm{gm}}(X,A)}(M, N \otimes_A P) \tag{3.2.7.1}$$

is an equivalence: this follows from the fact that P is a finite colimit of free A-modules. By Corollary 3.2.5 we can assume that Λ' is a ring of finite type over $\mathbb{Z}$, that is a quotient $\mathbb{Z}[T_1, \dots, T_r]/I$ of a polynomial ring by an ideal. Say we knew the result for $\Lambda' = \mathbb{Z}[T_1, \dots, T_r]$. Then as $\mathbb{Z}[T_1, \dots, T_r]/I$ is a perfect $\mathbb{Z}[T_1, \dots, T_r]$, the proof would be finished by (3.2.7.1). By induction it suffices to prove that if the result is known for a Noetherian ring A, then we can show that it holds for $A[T]$. But then writing

$$A[T] = \mathrm{colim}_n \, A[X]_{\leqslant n}$$

with $A[X]_{\leqslant n}$ the polynomials of degree $\leqslant n$, using (3.2.7.1) we see that is suffices to prove that for M, N in $\mathrm{DN}_{\mathrm{gm}}(X, A)$, the map

$$\mathrm{colim}_n \, \mathrm{map}_{\mathrm{DN}_{\mathrm{gm}}(X,A)}(M, N \otimes_A A[X]_{\leqslant n}) \to \mathrm{map}_{\mathrm{DN}_{\mathrm{gm}}(X,A)}(M, \mathrm{colim}_n \, N \otimes_A A[X]_{\leqslant n})$$

is an equivalence. Finally, this can be checked after tensorisation by $\mathbb{Q}$ (where this holds because M becomes compact), and after tensorisation by $\mathbb{Z}/p$ where this follows from [HRS23, Lemma 5.3] (note that *loc. cit.* is stated for a ring and not a derived ring, but the proof is the same for our derived ring $A \otimes_{\mathbb{Z}} \mathbb{Z}/p$: pushforwards commutes with uniformly bounded below colimits by [BGH20, Corollary 3.10.5] and then one has to prove the statement for the internal Hom, which is obvious is M is dualisable, and follows in general by dévissage.) ☐

Remark 3.2.8. The same proof gives the same result for $\mathrm{DM}^{\text{ét}}_{\text{gm}}$.

3.3 Compatibilities of the six functors and Verdier duality

We now study the compatibility of the Nori realisation with the six functors. We also show that the six functors induce functors on constructible motives when restricted to quasi-excellent schemes. Recall that by [AGV22, Theorem 4.6.1] this is also the case for $\mathrm{DM}^{\text{ét}}$.

Theorem 3.3.1. *Let Λ be a commutative ring. Over qcqs $\mathbb{Q}$-schemes of finite dimension, the Nori realisation:*

$$\rho_{\mathrm{N}} := - \otimes \mathscr{N} \colon \mathrm{DM}^{\text{ét}}(-,\Lambda) \to \mathrm{DN}(-,\Lambda)$$

is compatible with the operations of type f^*, f_*, $f_!$, $f^!$, $\otimes$ and $\underline{\mathrm{Hom}}(M,-)$ for M of geometric origin.

Proof. The theorem amounts to proving that the exchange transformations are equivalences, which can be tested after tensoring those maps with $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for any prime number p. After tensoring with $\mathbb{Z}/p\mathbb{Z}$, the Nori realisation becomes an equivalence so the exchange maps are also equivalences. After tensoring with $\mathbb{Q}$, those maps can be interpreted as the same maps but with coefficients $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ since the operations of type f^*, f_*, $f_!$, $f^!$ and $\underline{\mathrm{Hom}}(M,-)$ for M of geometric origin are compatible with rationalisation by Assertion 2. of Theorem 3.2.2. Hence, we can assume that Λ is a $\mathbb{Q}$-algebra.

The case of left adjoints is given by the definitions of the operations f^*, $f_!$ and $\otimes$ on $\mathrm{DN}(-,\Lambda)$. Now recall that if X is a qcqs $\mathbb{Q}$-scheme, we have

$$\mathrm{DN}(X,\Lambda) = \mathrm{DM}^{\text{ét}}(X,\Lambda) \otimes_{\mathrm{DM}^{\text{ét}}(\mathrm{Spec}(\mathbb{Q}),\Lambda)} \mathrm{DN}(\mathrm{Spec}(\mathbb{Q}),\Lambda)$$

and that the operation f^* on DN is defined as $f^* \otimes \mathrm{Id}_{\mathrm{DN}(\mathrm{Spec}(\mathbb{Q}),\Lambda)}$. Recall that $\mathrm{DM}^{\text{ét}}(\mathrm{Spec}(\mathbb{Q}),\Lambda)$ is compactly generated by [EHIK21b, Lemma 5.1] and that $\mathrm{DN}(\mathrm{Spec}(\mathbb{Q}),\Lambda)$ is also compactly generated by [Iwa18, Lemma 2.6]. Recall also that if $f \colon Y \to X$ is a morphism of schemes, the functor f^* preserves compact objects and therefore the functor f_* is colimit-preserving. We can therefore apply Proposition 1.1.1 to show that $f_* \otimes \mathrm{Id}_{\mathrm{DN}(\mathrm{Spec}(\mathbb{Q}),\Lambda)}$ is right adjoint to this functor which exactly means that f_* is compatible with the Nori realisation. The case of the functor $\underline{\mathrm{Hom}}(M,-)$ can be deduced by the same argument applied to the functor $M \otimes -$ because geometric objects are compact (as the generators are the images of compact étale motives). ☐

Proposition 3.3.2. *Let Λ be a commutative ring. The six functors preserve the category $\mathrm{DN}_{\mathrm{gm}}(-,\Lambda)$ when restricted to quasi-excellent finite-dimensional $\mathbb{Q}$-schemes and to morphisms of finite type between these schemes. More precisely*

1. *If X is a qcqs $\mathbb{Q}$-scheme, then $\otimes_{\mathrm{N}}$ induces a monoidal structure on $\mathrm{DN}_{\mathrm{gm}}(X,\Lambda)$. It is closed when X is quasi-excellent and finite-dimensional.*

2. *If f is a morphism between qcqs $\mathbb{Q}$-schemes, then f^* preserves geometric objects. If f is furthermore of finite type, then $f_!$ also preserves geometric objects.*

3. If f is a morphism of finite type between quasi-excellent finite-dimensional $\mathbb{Q}$-schemes, then f_* and $f^!$ preserve geometric objects.

Proof. The first part of (1), as well as (2) have already been proved in Theorem 3.2.2. For the rest of the proposition, we can work over quasi-excellent finite-dimensional $\mathbb{Q}$-schemes which are Noetherian by assumption, so that the results of [CD16] apply.

Let X be a quasi-excellent finite-dimensional $\mathbb{Q}$-scheme. The motives of type $\rho_{\mathrm{N}}(M)$ with M in $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\Lambda)$ generate $\mathrm{DN}_{\mathrm{gm}}(X,\Lambda)$ as a thick subcategory of $\mathrm{DN}(X,\Lambda)$. Therefore, the result follows from the same result for $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}$ which is [CD16, Corollary 6.2.14] and from the compatibility of the six functors with the Nori realisation which is Theorem 3.3.1. ☐

Proposition 3.3.3. *Let Λ be a commutative ring. Consider an excellent scheme B of dimension 2 or less as well as a regular separated B-scheme of finite type S, endowed with a geometric and $\otimes$-invertible object ω_S in $\mathrm{DN}(S,\Lambda)$. For any separated morphism of finite type $f\colon X \to S$, we define the* Verdier duality functor $\mathbb{D}_X$ *by the formula*

$$\mathbb{D}_X(M) := \underline{\mathrm{Hom}}_{\mathrm{N}}(M, f^!(\omega_S)).$$

Then, for any geometric Nori motive M, the canonical map $M \to \mathbb{D}_X(\mathbb{D}_X(M))$ is an isomorphism.

Proof. From the compatibility of the Nori realisation with the six functors (Theorem 3.3.1) and the same result for étale motives [CD16, Theorem 6.2.17], we deduce the result when $\omega_S = \Lambda_S$. This implies the result in general using [CD19, Proposition 4.4.22]. ☐

Remark 3.3.4. A priori with rational coefficients we have two Verdier duality functors, at least rationally: the one constructed using the above formula, and the one obtained using the universal property of Nori motives. Thankfully, these two functors are canonically isomorphic by Terenzi's [Ter24, Proposition 5.19]. The proof uses the existence of a Nori realisation of rational étale motives.

3.4 The ℓ-adic realisation

In this paragraph, we define the ℓ-adic realisation on Nori motives.

Proposition 3.4.1. *Let Λ be a commutative ring, let X be a qcqs $\mathbb{Q}$-scheme and let ℓ be a prime number. The maps*

$$\mathrm{D}(X_{\text{ét}},\Lambda)_{\mathrm{tors}} \xrightarrow{\rho_!} \mathrm{DM}^{\text{ét}}(X,\Lambda)_{\mathrm{tors}} \xrightarrow{\rho_{\mathrm{N}}} \mathrm{DN}(X,\Lambda)_{\mathrm{tors}}$$

$$\mathrm{D}(X_{\text{ét}},\Lambda)^{\wedge}_{\ell} \xrightarrow{(\rho_!)^{\wedge}_{\ell}} \mathrm{DM}^{\text{ét}}(X,\Lambda)^{\wedge}_{\ell} \xrightarrow{(\rho_{\mathrm{N}})^{\wedge}_{\ell}} \mathrm{DN}(X,\Lambda)^{\wedge}_{\ell}$$

are equivalences.

Proof. The second row of equivalences follows from the first and from the identification of p-torsion objects with p-complete objects. By Lemma 1.3.1, it suffices to prove the first identification after tensoring with $\mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}}$. The proposition then follows from the rigidity theorem [BH21, Theorem 3.1] and from the fact that the canonical map $\mathbb{1}_X/n \to \mathscr{N}_X/n$ is an equivalence. ☐

Definition 3.4.2. Let X be a qcqs $\mathbb{Q}$-scheme. The big ℓ-adic realisation of Nori motives is the functor

$$R^{\mathrm{big}}_{\ell}\colon \mathrm{DN}(X,\mathbb{Z}) \xrightarrow{(-)^{\wedge}_{\ell}} \mathrm{DN}(X,\mathbb{Z})^{\wedge}_{\ell} \xrightarrow{\sim} \mathrm{D}(X_{\text{ét}},\mathbb{Z})^{\wedge}_{\ell}.$$

Remark 3.4.3. As in [CD16, Section 7.2], we can endow $\mathrm{D}(-_{\text{ét}},\mathbb{Z})^{\wedge}_{\ell}$ with the six functors formalism and show that the big ℓ-adic realisation is compatible with the six functors.

We now construct the ℓ-adic realisation for constructible motives. It will land into the category of constructible sheaves of [BS15, HRS23].

Construction 3.4.4. If X is a qcqs finite-dimensional $\mathbb{Q}$-scheme, we have maps:

$$\mathrm{DN}(X,\mathbb{Z}) \xrightarrow{R_\ell^{\mathrm{big}}} \mathrm{D}(X_{\mathrm{\acute{e}t}},\mathbb{Z})^\wedge_\ell \xrightarrow{-\otimes\mathbb{Z}/\ell^n\mathbb{Z}} \mathrm{D}(X_{\mathrm{\acute{e}t}},\mathbb{Z}/\ell^n\mathbb{Z}).$$

The image of $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ through the latter is contained in the subcategory of constructible sheaves $\mathrm{D}_{\mathrm{cons}}(X_{\mathrm{\acute{e}t}},\mathbb{Z}/\ell^n\mathbb{Z})$ (that is those complexes whose restriction to any open affine subset are dualisable on a finite stratification). The change of sites from the étale to the pro-étale site then induces an equivalence

$$\mathrm{D}_{\mathrm{cons}}(X_{\mathrm{\acute{e}t}},\mathbb{Z}/\ell^n\mathbb{Z}) \to \mathrm{D}_{\mathrm{cons}}(X_{\mathrm{pro\acute{e}t}},\mathbb{Z}/\ell^n\mathbb{Z})$$

using [HRS23, Proposition 7.1]. We therefore get a map

$$\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \to \lim \mathrm{D}_{\mathrm{cons}}(X_{\mathrm{\acute{e}t}},\mathbb{Z}/\ell^n\mathbb{Z}).$$

[HRS23, Proposition 5.1] ensures that the right-hand side is equivalent to $\mathrm{D}_{\mathrm{cons}}(X_{\mathrm{pro\acute{e}t}},\mathbb{Z}_\ell)$ the ∞-category of constructible proétale sheaves (again, those complexes whose restriction to any open affine subset are dualisable on a finite stratification but in the pro-étale world). The categories $\mathrm{D}_{\mathrm{cons}}((-)_{\mathrm{pro\acute{e}t}},\mathbb{Z}_\ell)$ are endowed with the six functors over quasi-excellent base schemes using [BS15, Section 6.7]; these are defined by passing to the limit from the functors over $\mathrm{D}_{\mathrm{cons}}((-)_{\mathrm{\acute{e}t}},\mathbb{Z}/\ell^n\mathbb{Z})$ which are induced from those over $\mathrm{D}(-_{\mathrm{\acute{e}t}},\mathbb{Z}/\ell^n\mathbb{Z})$. This yields a map

$$R_\ell\colon \mathrm{DN}_{\mathrm{gm}}(-,\mathbb{Z}) \to \mathrm{D}_{\mathrm{cons}}((-)_{\mathrm{\acute{e}t}},\mathbb{Z}_\ell)$$

compatible with the six functors over quasi-excellent schemes which we call the *ℓ-adic realisation.*

Finally, if Λ is a localisation of a ring integral and flat over $\mathbb{Z}$, we can tensor the above functor by Perf_Λ over $\mathrm{Perf}_\mathbb{Z}$ to get the ℓ-adic realisation

$$R_\ell\colon \mathrm{DN}_{\mathrm{gm}}(-,\Lambda) \to \mathrm{D}_{\mathrm{cons}}((-)_{\mathrm{pro\acute{e}t}},\mathbb{Z}_\ell)\otimes_{\mathrm{Perf}_\mathbb{Z}}\mathrm{Perf}_\Lambda \xrightarrow{\sim} \mathrm{D}_{\mathrm{cons}}((-)_{\mathrm{pro\acute{e}t}},\Lambda_\ell),$$

where the last equivalence is [HS23, Corollary 2.3].

This functor is compatible with the functors of type $\otimes$, f^* and $f_!$ and when we restrict to quasi-excellent base schemes and to finite type morphisms, it is compatible with the functors of type f_* and $f^!$. Indeed, the same is true for the ℓ-adic realisation of étale motives and therefore this follows from Theorem 3.3.1.

Proposition 3.4.5. *Let Λ be a localisation of a flat integral extension of $\mathbb{Z}$. Let X be a qcqs $\mathbb{Q}$-scheme of finite dimension.*

1. *If Λ is a $\mathbb{Q}$-algebra, and ℓ is a prime number, then the ℓ-adic realisation functor*

$$R_\ell\colon \mathrm{DN}_{\mathrm{gm}}(X,\Lambda) \to \mathrm{D}_{\mathrm{cons}}(X_{\mathrm{pro\acute{e}t}},\Lambda_\ell)$$

is conservative.

2. *Otherwise, the family of the R_ℓ where $\ell \notin \Lambda^\times$ is conservative.*

Proof. We first show the first assertion. By continuity (Theorem 3.2.2), we can reduce to the case where X is a finite type $\mathbb{Q}$-scheme. As Λ is a $\mathbb{Q}$-algebra, we have by [Tub23, Corollary 2.19] an equivalence

$$\mathrm{DN}_{\mathrm{gm}}(X,\Lambda) \xrightarrow{\sim} \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Q}))\otimes_{\mathrm{Perf}_\mathbb{Q}}\mathrm{Perf}_\Lambda.$$

Moreover using the description of mapping spectra in tensor products provided by [HL23, Proposition 3.5.5] we see that if the functor

$$R_\ell \colon \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Q}) \to \mathrm{D}_{\mathrm{cons}}(X_{\text{proét}}, \mathbb{Q}_\ell)$$

is conservative, so is the functor of the proposition. Thus this follows from [IM24, Proposition 6.11].

We now prove the second assertion. As the family $\{- \otimes \mathbb{Q}, - \otimes \mathbb{Z}/\ell\mathbb{Z} \mid \ell \notin \Lambda^\times\}$ is conservative, it suffices to show that if $R_\ell(M/\ell)$ vanishes, then so does M/ℓ for M a geometric motive. This amounts to the rigidity theorem. ☐

We finish by a conjectural aspect of our construction: let X be a finite-dimensional qcqs $\mathbb{Q}$-scheme and let Λ be a commutative ring. By construction we have a functor

$$\rho_{\mathrm{N}} \colon \mathrm{DM}^{\text{ét}}(X, \Lambda) \to \mathrm{DN}(X, \Lambda).$$

Proposition 3.4.6. *Assume that for all fields K of finite transcendence degree over $\mathbb{Q}$, the ∞-category $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(K, \mathbb{Q})$ is endowed with a motivic t-structure. Then for X as above, the functor ρ_{N} is an equivalence.*

Proof. First, note that the functor ρ_{N} is the tensorisation by Mod_Λ of the functor with coefficients $\mathbb{Z}$, so that it suffices to deal with this case. Moreover as this functor is an equivalence after being tensored by $\mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}}$ for all $n \in \mathbb{N}^*$, it suffices to check that it becomes an equivalence after being tensored by $\mathrm{Mod}_{\mathbb{Q}}$. As $\mathscr{N}_X$ is pulled back from $\mathrm{Spec}(\mathbb{Q})$, it also suffices to prove that ρ_{N} is an equivalence over $\mathrm{Spec}(\mathbb{Q})$. The result then follows from [Tub23, Theorem 4.11]. ☐

Remark 3.4.7. The hypothesis in the previous proposition could be weakened. Indeed it is very possible that using Pridham's [Pri20, Section 2.5], one can prove that if $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(\mathrm{Spec}(\mathbb{Q}), \mathbb{Q})$ has a motivic t-structure, then the functor

$$\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(\mathrm{Spec}(\mathbb{Q}), \mathbb{Q}) \to \mathrm{D}(\mathrm{Ind}\, \mathcal{M}(\mathbb{Q}, \mathbb{Q}))$$

is fully faithful, so that, because its image is $\mathrm{D}^b(\mathcal{M}(\mathbb{Q}, \mathbb{Q}))$ (because $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$ is Noetherian, see [Hub93]), this induces an equivalence $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(\mathrm{Spec}(\mathbb{Q}), \mathbb{Q}) \simeq \mathrm{D}^b(\mathcal{M}(\mathbb{Q}, \mathbb{Q}))$, thus the functor

$$\mathrm{DM}^{\text{ét}}(\mathrm{Spec}(\mathbb{Q}), \mathbb{Q}) \to \mathrm{DN}(\mathrm{Spec}(\mathbb{Q}), \mathbb{Q})$$

is an equivalence, meaning that the map of rings $\mathbb{1}_k \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathscr{N}_{k,\mathbb{Q}}$ is an equivalence. In other terms, to prove that $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \Lambda)$ has a motivic t-structure for all X as above and Λ, it "suffices" to prove that $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(\mathrm{Spec}(\mathbb{Q}), \mathbb{Q})$ has a motivic t-structure.

4 The abelian category of ordinary Nori motives.

4.1 Relative Nori motives as a pullback

Until further notice, k is an algebraically closed subfield of $\mathbb{C}$. For each finite type k-scheme X, Ayoub constructed in [Ayo10] a Betti realisation functor

$$\rho_{\mathrm{B}} \colon \mathrm{DM}^{\text{ét}}(X, \Lambda) \to \mathrm{D}(X^{\mathrm{an}}, \mathbb{Z})$$

to the derived category of analytic sheaves of abelian groups on X^{an}. Let $\mathscr{B}_k$ be the algebra in $\mathrm{DM}^{\text{ét}}(k, \mathbb{Z})$ representing Betti cohomology [Ayo22, Notation 1.57] and if X is a finite type k-scheme, let $\mathscr{B}_X$ be the pullback of $\mathscr{B}_k$ to X. Ayoub considered the functor on finite type k-schemes

$$\mathrm{Mod}_{\mathscr{B}_{(-)}}(\mathrm{DM}^{\text{ét}}(-, \mathbb{Z}))\colon \left(\mathrm{Sch}_k^{ft}\right)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathbb{Z}})$$

of which he gave an explicit description in [Ayo22, Theorem 1.93]: he proved that the tautological functor $\mathrm{DM}^{\text{ét}}(-, \mathbb{Z}) \to \mathrm{Mod}_{\mathscr{B}_{(-)}}(\mathrm{DM}^{\text{ét}}(-, \mathbb{Z}))$ factors the refined Betti realisation

$$\widetilde{\rho}_{\mathrm{B}}\colon \mathrm{DM}^{\text{ét}}(-, \mathbb{Z}) \to \mathrm{Ind}\,\mathrm{D}_{\mathrm{cons}}((-)^{\mathrm{an}}, \mathbb{Z})$$

where $\mathrm{D}_{\mathrm{cons}}((-)^{\mathrm{an}}, \mathbb{Z})$ denotes the category of constructible analytic complexes, and that for any finite type k-scheme X, the induced functor $\mathrm{Mod}_{\mathscr{B}_X}(\mathrm{DM}^{\text{ét}}(X, \mathbb{Z})) \to \mathrm{Ind}\,\mathrm{D}_{\mathrm{cons}}((-)^{\mathrm{an}}, \mathbb{Z})$ is fully faithful. Its essential image is furthermore compactly generated: it is the indisation of a full subcategory $\mathrm{D}_{\mathrm{gm}}(X^{\mathrm{an}}, \mathbb{Z})$ of $\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{an}}, \mathbb{Z})$ of complexes of *geometric origin.*

The realisation to Nori motives

$$\rho_{\mathrm{N}}\colon \mathrm{DM}^{\text{ét}}(-, \mathbb{Q}) \to \mathrm{DN}(-, \mathbb{Q})$$

factors the refined Betti realisation of étale motives, hence there exists a map of algebra

$$\varphi_X\colon \mathscr{N}_{X,\mathbb{Q}} \to \mathscr{B}_{X,\mathbb{Q}}$$

compatible with pullback by finite type morphisms.

Proposition 4.1.1. *There exist a natural map of algebras $\mathscr{N}_X \to \mathscr{B}_X$ giving back φ_X after rationalisation. Moreover, it induces a pullback diagram*

$$\begin{tikzcd}
\mathscr{N}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{B}_X \arrow[d] \\
\mathscr{N}_{X,\mathbb{Q}} \arrow[r] & \mathscr{B}_{X,\mathbb{Q}}
\end{tikzcd}.$$

Proof. Let A be the pullback

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{B}_X \arrow[d] \\
\mathscr{N}_{X,\mathbb{Q}} \arrow[r] & \mathscr{B}_{X,\mathbb{Q}}
\end{tikzcd}.$$

Thanks to Artin's Comparison Theorem [SGA4, Exposé XI, Théorème 4.4 & Exposé XVI, Théorème 4.1] (see also Step 1 of the proof of [Ayo22, Lemma 1.107]), the Hasse arithmetic fracture square for $\mathscr{B}_X$ is of the form:

$$\begin{tikzcd}
\mathscr{B}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim_n \mathbb{1}_X/n \arrow[d] \\
\mathscr{B}_{X,\mathbb{Q}} \arrow[r] & (\lim_n \mathbb{1}_X/n) \otimes \mathbb{Q}
\end{tikzcd},$$

so that by pasting pullbacks, A also fits in the pullback square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim_n \mathbb{1}_X/n \arrow[d] \\
\mathscr{N}_{X,\mathbb{Q}} \arrow[r] & (\lim_n \mathbb{1}_X/n) \otimes \mathbb{Q}
\end{tikzcd}.$$

where we recognise the Hasse fracture square of $\mathscr{N}_{\mathbb{Z}}$ (the arrow are the correct ones because the equivalence $\mathscr{N}_X/n \xrightarrow{\sim} \mathscr{B}_X/n$ factors the equivalence $\mathbb{1}_X/n \xrightarrow{\sim} \mathscr{B}_X/n$.) □

Therefore we have a canonical functor

$$\mathrm{DN}(-,\mathbb{Z}) \to \operatorname{Ind}\mathrm{D}_{\mathrm{gm}}((-)^{\mathrm{an}},\mathbb{Z}).$$

Corollary 4.1.2. *Relative Nori motives with integral coefficients (even relative to a non algebraically closed field) have a Betti realisation*

$$\mathrm{DN}(-,\mathbb{Z}) \to \mathrm{D}((-)^{\mathrm{an}},\mathbb{Z}).$$

which is compatible with the six functors.

Proposition 4.1.3. *The square*

$$\begin{tikzcd}
\mathrm{DN}(-,\mathbb{Z}) \arrow[r] \arrow[d] & \operatorname{Ind}\mathrm{D}_{\mathrm{gm}}((-)^{\mathrm{an}},\mathbb{Z}) \arrow[d] \\
\mathrm{DN}(-,\mathbb{Q}) \arrow[r] & \operatorname{Ind}\mathrm{D}_{\mathrm{gm}}((-)^{\mathrm{an}},\mathbb{Q})
\end{tikzcd}$$

is cartesian in $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}},\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathbb{Z}}))$.

Proof. It suffices to check the proposition after evaluating at a finite type k-scheme X. If we let $P(X)$ be the pullback of the proposition, there is a functor $\mathrm{DN}(X,\mathbb{Z}) \to P(X)$. This functor preserves compact objects (note that k is algebraically closed thus $\mathrm{DN}(X,\mathbb{Z}) \simeq \operatorname{Ind}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ by Theorem 3.2.2), thus its right adjoint commutes with colimits, whence commutes with $-\otimes\mathbb{Q}$ so that we may use Lemma 1.3.1. ☐

Corollary 4.1.4. *Let k be any subfield of $\mathbb{C}$ and let X be a finite type k-scheme. The Betti realisation*

$$\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \to \mathrm{D}^b_c(X^{\mathrm{an}},\mathbb{Z})$$

is conservative.

Proof. Indeed, the functor $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \to \mathrm{DN}_{\mathrm{gm}}(X\times_k \overline{k},\mathbb{Z})$ is conservative by continuity and Galois descent (if $M_{\overline{k}} = 0$, there exists a finite extension L of k such that M_L is zero, and then by Galois descent the functor $(-)_L$ is conservative, thus $M = 0$). Then by Proposition 4.1.3 the functor $\mathrm{DN}_{\mathrm{gm}}(X_{\overline{k}},\mathbb{Z}) \to \mathrm{D}^b_c(X^{\mathrm{an}},\mathbb{Z})$ is conservative: if $R_{\mathrm{B}}(M) = 0$, then by conservativity of the realisation rationally we have that $M\otimes\mathbb{Q} = 0$, thus both $R_{\mathrm{B}}(M)$ and $M\otimes\mathbb{Q}$ are zero thus $M = 0$ in the pullback. ☐

The pullback square of Proposition 4.1.3 allows us to endow the $\mathrm{DN}(X,\mathbb{Z})$ with a t-structure. Recall first that if Λ is regular, the category $\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{an}},\Lambda)$ is endowed with a t-structure induced by the t-structure of $\mathrm{D}(X^{\mathrm{an}},\Lambda)$. Lurie's [SAG, Lemma C.2.4.3] ensures that $\operatorname{Ind}\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{an}},\Lambda)$ is endowed with a t-structure whose negative (resp. positive) objects are the filtered colimits of negative (resp. positive) objects, and that therefore, the canonical functor

$$\operatorname{Ind}\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{an}},\Lambda) \to \mathrm{D}(X^{\mathrm{an}},\Lambda)$$

is t-exact. One consequence of [Ayo22, Theorem 1.93], is that $\mathrm{D}_{\mathrm{gm}}(X^{\mathrm{an}},\Lambda)$ has a t-structure induced by that of $\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{an}},\Lambda)$; we will denote its heart by $\mathrm{Sh}_{\mathrm{gm}}(X^{\mathrm{an}},\Lambda)$. Using [SAG, Lemma C.2.4.3] again, the stable ∞-category $\operatorname{Ind}\mathrm{D}_{\mathrm{gm}}((-)^{\mathrm{an}},\Lambda)$ has a t-structure which we call the *ordinary t-structure.*

On the other hand, we have $\mathrm{DN}(X,\mathbb{Q}) = \operatorname{Ind}\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Q}))$. In [Tub23, Definition 2.1], the second author defined a t-structure on $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Q}))$ that is called the *constructible t-structure* in the quoted paper but that we will call the *ordinary t-structure* in this paper for consistency. Denote its heart by $\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Q})$, then by [Tub23, Corollary 2.19], the natural map

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Q})) \to \mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Q}))$$

is an equivalence. Furthermore, the functor $\mathrm{D}^b(\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Q})) \to \mathrm{D}_{\mathrm{gm}}(X^{\mathrm{an}},\mathbb{Q})$ is t-exact with the ordinary t-structure on both sides. Therefore, using Lemma 1.2.1, we get

Proposition 4.1.5. *Assume that k is an algebraically closed subfield of $\mathbb{C}$. Let X be a finite type k-scheme, there is a t-structure on $\mathrm{DN}(X,\mathbb{Z})$ such that the rationalisation, endowed with its ordinary t-structure, and the Betti realisation are t-exact. Furthermore, we have a pullback square:*

$$\begin{tikzcd}
\mathrm{DN}(X,\mathbb{Z})^\heartsuit \arrow[r] \arrow[d] & \operatorname{Ind}\mathrm{Sh}_{\mathrm{gm}}(X^{\mathrm{an}},\mathbb{Z}) \arrow[d] \\
\operatorname{Ind}\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Q}) \arrow[r] & \operatorname{Ind}\mathrm{Sh}_{\mathrm{gm}}(X^{\mathrm{an}},\mathbb{Q}).
\end{tikzcd}$$

Remark 4.1.6. Using the ideas of this section, we can define a category of Nori motivic spectra which is akin to the category $\mathrm{SH}^{\text{ét}}(X)$ of étale motivic spectra when X is a scheme of finite type over an *algebraically closed* field of characteristic 0. First, let $\mathscr{B}_X^{\mathbb{S}}$ be the algebra in $\mathrm{SH}^{\text{ét}}(X)$ representing Betti cohomology with $\mathbb{S}$-coefficients. Once again, Ayoub constructed in [Ayo22, Theorem 1.93] an equivalence $\mathrm{Mod}_{\mathscr{B}_X^{\mathbb{S}}}(\mathrm{SH}^{\text{ét}}(X)) \to \operatorname{Ind}\mathrm{D}_{\mathrm{gm}}(X^{\mathrm{an}},\mathbb{S})$, where $\mathrm{D}_{\mathrm{gm}}(X^{\mathrm{an}},\mathbb{S})$ is the full subcategory of the category $\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{an}},\mathbb{S})$ of constructible analytic spectral sheaves, made of objects of geometric origin and is endowed with an induced t-structure.

By analogy with Proposition 4.1.1, we can define $\mathscr{N}_X^{\mathbb{S}}$ to fit in the pullback

$$\begin{tikzcd}
\mathscr{N}_X^{\mathbb{S}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{B}_X^{\mathbb{S}} \arrow[d] \\
\mathscr{N}_{X,\mathbb{Q}} \arrow[r] & \mathscr{B}_{X,\mathbb{Q}}
\end{tikzcd}$$

and then define $\mathrm{SH}^{\text{ét}}_{\mathrm{Nori}}(X) := \mathrm{Mod}_{\mathscr{N}_X^{\mathbb{S}}}(\mathrm{SH}^{\text{ét}}(X))$. We now claim that the square

$$\begin{tikzcd}
\mathrm{SH}^{\text{ét}}_{\mathrm{Nori}}(X) \arrow[r] \arrow[d] & \operatorname{Ind}\mathrm{D}_{\mathrm{gm}}(X^{\mathrm{an}},\mathbb{S}) \arrow[d] \\
\mathrm{DN}(-,\mathbb{Q}) \arrow[r] & \operatorname{Ind}\mathrm{D}_{\mathrm{gm}}((-)^{\mathrm{an}},\mathbb{Q})
\end{tikzcd}$$

is cartesian. The full faithfulness of the map $\Psi_X\colon \mathrm{SH}^{\text{ét}}_{\mathrm{Nori}}(X) \to P(X)$ (where $P(X)$ denotes the pullback) follows from the general claim that if $A = B \times_D C$ is a pullback of commutative algebra objects in a stable ∞-category $\mathcal{C}$, the map $\mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Mod}_B(\mathcal{C}) \times_{\mathrm{Mod}_D(\mathcal{C})} \mathrm{Mod}_C(\mathcal{C})$ is fully faithful which can be seen using a direct computation of mapping spectra. To show that the map Ψ_X is essentially surjective, note that $P(X)$ is endowed with an ordinary t-structure by Lemma 1.2.1 so that it suffices to reach the objects of its heart. We claim that it is the case. Indeed the functor

$$-\otimes_{\mathbb{S}}\mathbb{Z}\colon \operatorname{Ind}\mathrm{D}_{\mathrm{gm}}(X^{\mathrm{an}},\mathbb{S}) \to \operatorname{Ind}\mathrm{D}_{\mathrm{gm}}(X^{\mathrm{an}},\mathbb{Z})$$

yields the equivalence $\tau^{\geqslant 0}(-\otimes_{\mathbb{S}}\mathbb{Z})$ between the hearts, therefore we have an equivalence $P(X)^{\heartsuit} \to \mathrm{DN}(X,\mathbb{Z})^{\heartsuit}$. Using that at the generic point η for a projective smooth morphism f, the object $f_*\mathbb{Z}$ is the sum in $\mathrm{DN}(\eta,\mathbb{Z})$ of its cohomology objects (by Deligne's criterion for degeneracy of spectral sequences [Del68, Théorème 1.5]), one can prove the claim at the generic point, and then localisation enables one to prove that Ψ_X is an equivalence in general. Hence we have an ordinary motivic t-structure on $\mathrm{SH}^{\text{ét}}_{\text{Nori}}(X)$ whose heart is the same as that of $\mathrm{DN}(X,\mathbb{Z})$.

4.2 The ordinary t-structure on geometric Nori motives

Let X be a finite dimensional qcqs $\mathbb{Q}$-scheme. In this section we construct an ordinary t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$. Note that by Corollary 2.3.7 we already have a t-structure on $\mathrm{DN}_{\mathrm{gm}}(k,\mathbb{Z})$ for any subfield k of $\mathbb{C}$. By continuity and because a filtered colimit of stable ∞-categories with t-structures and t-exact functors has a natural t-structure, we can assume that X is a finite type $\mathbb{Q}$-scheme. In this case we may use the punctual gluing of t-structures due to Vaish [Vai20, Proposition 3.8]. In order to use his result, we have to verify the condition he calls "continuity of the t-structure", which is the purpose of the next proposition. Let us first recall how the t-structure is defined:

Definition 4.2.1. For each $x \in X$, denote by $i_x\colon \mathrm{Spec}(\kappa(x)) \to X$ the inclusion of x in X. An object $M \in \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ is called

1. *punctually connective* if for each $x \in X$ we have $i_x^*M \in \mathrm{D}^b(\mathcal{M}(\kappa(x),\mathbb{Z}))^{\leqslant 0}$
2. *punctually coconnective* if for each $x \in X$ we have $i_x^!M \in \mathrm{D}^b(\mathcal{M}(\kappa(x),\mathbb{Z}))^{\geqslant 1}$.

We denote by $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\leqslant 0}$ and $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\geqslant 1}$ the associated full subcategories.

Proposition 4.2.2. *Let X be a finite type $\mathbb{Q}$-scheme. Let $x \in X$ and let $M_x \in \mathrm{D}^b(\mathcal{M}(\kappa(x),\mathbb{Z}))$.*

1. *If $M \in \mathrm{D}^b(\mathcal{M}(\kappa(x),\mathbb{Z}))^{\leqslant 0}$ there exists a nonempty open subset U of $\overline{\{x\}}$ and an object $M \in \mathrm{DN}_{\mathrm{gm}}(U,\mathbb{Z})^{\leqslant 0}$ such that $i_x^*M = M_x$.*
2. *If $M \in \mathrm{D}^b(\mathcal{M}(\kappa(x),\mathbb{Z}))^{\geqslant 1}$ there exists a nonempty open subset U of $\overline{\{x\}}$ and an object $M \in \mathrm{DN}_{\mathrm{gm}}(U,\mathbb{Z})^{\geqslant 1}$ such that $i_x^!M = M_x$.*

Proof. In both cases we can assume that x is a generic point of X and that X is irreducible, and even smooth up to replace X by one of its dense open subsets. In this case both i_x^* and $i_x^!$ agree. Thus the proof of the two cases are almost the same and we only deal with the first point. By continuity, we know that there exist some $U_0 \subset X$ nonempty and open, and $M_0 \in \mathrm{DN}_{\mathrm{gm}}(U_0,\mathbb{Z})$ such that $i_x^*M_0 = M_x$. Now the Betti realisation being conservative, we know that there exists some integer $C \in \mathbb{N}^*$ such that if p is a prime verifying that the multiplication by p is zero on M_0, then $p \mid C$. Moreover, there is some other integer $a > 0$ such that the Betti realisation of M_0 lives in degrees $[-a,a]$. Because $\mathbb{Z}_\ell$ is flat over $\mathbb{Z}$ and the functor $\mathrm{D}^b_c(X_{\text{ét}},\mathbb{Z}_\ell) \to \mathrm{D}^b_c(X^{\mathrm{an}},\mathbb{Z}_\ell)$ are conservative and t-exact, we see that for each prime number ℓ, the ℓ-adic realisation of M_0 is concentrated in degrees $[-a,a]$. Let U_1 be a nonempty open subset of U_0 on which M_0 is dualisable. For each $j \in [1,a]$ and each prime number ℓ dividing C, because $i_x^*\mathrm{H}^j(R_\ell(M_1))$ is zero and $\mathrm{H}^j(R_\ell(M_1))$ is a local system on $(U_1)_{\text{ét}}$, there exists a nonempty open subset $U_{j,\ell}$ of U_1 such that the restriction to $U_{j,\ell}$ of $\mathrm{H}^j(R_\ell(M_1))$ is zero. We let U be the intersection of all of those $U_{j,\ell}$ and M be the restriction of M_1 to U. Then we see that $R_{\mathrm{B}}(M)$ is connective, thus M is punctually connective, as wanted. $\square$

Theorem 4.2.3. *The conditions of Definition 4.2.1 define a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ and the Betti realisation is t-exact. Moreover, for any finite dimensional qcqs $\mathbb{Q}$-scheme X this induces a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ such that all ℓ-adic realisations are t-exact.*

Proof. The t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ is given by the above proposition together with [Vai20, Proposition 3.8]. The fact that the Betti realisation is t-exact can be checked on stalked at closed points, where it is true. Thus the ℓ-adic realisation are also t-exact, and when we endow $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$, for X a general finite dimentional qcqs $\mathbb{Q}$-scheme by continuity using Lemma 1.2.2, the ℓ-adic realisation stay t-exact because if $X = \lim X_i$ is a cofiltered limit with affine transitions, although the functor

$$\mathrm{colim}_i\, \mathrm{D}_{\mathrm{cons}}(X_{i,\mathrm{pro\acute{e}t}},\mathbb{Z}_\ell) \to \mathrm{D}_{\mathrm{cons}}(X_{\mathrm{pro\acute{e}t}},\mathbb{Z}_\ell)$$

is not an equivalence, it is t-exact. □

Definition 4.2.4. Let X be a finite dimentional qcqs $\mathbb{Q}$-scheme. The category of *ordinary Nori motives* $\mathcal{M}_{\mathrm{ord}}(X,\mathbb{Z})$ is the heart of the t-structure given by Theorem 4.2.3. We call the latter the *ordinary t-structure*.

Remark 4.2.5. All the construction above would work if we replace $\mathbb{Z}$ by a localisation of the ring of integers of a number field, finite or not over $\mathbb{Q}$. This is because in this case the category $\mathcal{M}(k,\Lambda)$ is well-behaved (because Λ is a Dedekind ring), so that Theorem 2.3.6 and Corollary 2.3.7 work in that case. For example, we have a t-structure with $\overline{\mathbb{Z}}$-coefficients.

Remark 4.2.6. As the ℓ-adic realisation is also conservative by Proposition 3.4.5, any t-exactness property of the ordinary t-structure on ℓ-adic sheaves transfers to the ordinary t-structure on Nori motives.

Remark 4.2.7. Let k be a subfield of $\mathbb{C}$ and let X be a k-variety. Arapura considered in [Ara13] an abelian category of motivic sheaves modelled on constructible sheaves. By construction, his category $\mathcal{M}_{\mathrm{Arapura}}(X,\mathbb{Z})$ has an universal property, so that there exists a faithful exact functor $\mathcal{M}_{\mathrm{Arapura}}(X,\mathbb{Z}) \to \mathcal{M}_{\mathrm{ord}}(X,\mathbb{Z})$ compatible with the Betti realisations. We do not know much about this functor, except that it is an equivalence for $X = \mathrm{Spec}(k)$.

4.3 The derived category of ordinary Nori motives

We now prove that $\mathrm{DN}_{\mathrm{gm}}$ is the derived category of ordinary Nori motives following Nori [Nor02]. To that end, we first need:

Lemma 4.3.1. *The functor $\mathrm{D}^b(\mathcal{M}_{\mathrm{ord}}(-,\Lambda))$ is a finitary étale hypersheaf on qcqs finite-dimensional schemes.*

Proof. The finitarity of the functor follows from Lemma 1.2.3 and the continuity of $\mathrm{DN}_{\mathrm{gm}}$. Because for f an étale map, the functors f^* and $f_!$ are t-exact, they exist on $\mathrm{D}^b(\mathcal{M}_{\mathrm{ord}}(-,\Lambda))$ as an adjunction, compatible with the realisation functor

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{ord}}(-,\Lambda)) \to \mathrm{DN}_{\mathrm{gm}}(-,\Lambda)$$

which is conservative. Thus, because $\mathrm{DN}_{\mathrm{gm}}(-,\Lambda)$ is an étale hypersheaf, the functor $\mathcal{M}_{\mathrm{ord}}(-,\Lambda)$ is an étale sheaf: it is a sub presheaf of $\mathrm{DN}_{\mathrm{gm}}(-,\Lambda)$ and the cohomological amplitude with respect to the ordinary t-structure can be tested étale-locally. We can now apply the same proof as [Tub23, Theorem 3.24] to conclude. □

Theorem 4.3.2. *Let Λ be a regular ring which is flat over $\mathbb{Z}$. Let X be a qcqs finite-dimensional $\mathbb{Q}$-scheme. Then, the canonical functor*

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{ord}}(X,\Lambda)) \to \mathrm{DN}_{\mathrm{gm}}(X,\Lambda)$$

is an equivalence.

Proof. By Lemma 4.3.1, continuity and Zariski hyperdescent for $\mathrm{DN}_{\mathrm{gm}}(-,\Lambda)$, we can assume that X is of finite type over $\mathbb{Q}$ and separated. Note that by [Bei87, Lemma 1.4], to prove the theorem, it is sufficient to prove that for any pair of Nori motives M, N in $\mathcal{M}_{\mathrm{ord}}(X,\Lambda)$ and any integer $i > 0$, the map

$$\mathrm{Ext}^i_{\mathcal{M}_{\mathrm{ord}}(X,\Lambda)}(M,N) = \mathrm{Hom}_{\mathrm{D}^b(\mathcal{M}_{\mathrm{ord}}(X,\Lambda))}(M,N[i]) \to \mathrm{Hom}_{\mathrm{DN}_{\mathrm{gm}}(X,\Lambda)}(M,N[i])$$

is an isomorphism, which, given the characterisation of the Ext^i, is equivalent to the functor

$$\mathrm{E}^i_M := \mathrm{Hom}_{\mathrm{DN}_{\mathrm{gm}}(X,\Lambda)}(M,(-)[i])\colon \mathcal{M}_{\mathrm{ord}}(X,\Lambda) \to \mathrm{Ab}$$

being effaceable for each $i > 0$ and each M in $\mathcal{M}_{\mathrm{ord}}(X,\Lambda)$. More explicitly, this means that for every Nori motive N in $\mathcal{M}_{\mathrm{ord}}(X,\Lambda)$, there exists a monomorphism $f\colon N \to P$ whose image under our functor E^i_M vanishes.

The proof of this fact is then very similar to the proof the second author gave, following Nori, in the case of Nori motives with rational coefficients (see [Nor02, Tub23]). Until the end of this proof, we will call *admissible* an Nori motive M in $\mathcal{M}_{\mathrm{ord}}(X,\Lambda)$ such that for all $i > 0$ the functor E^i_M is effaceable.

By remarking that all the proofs that lead to [Tub23, Corollary 2.16] work verbatim with integral coefficients (because we only use the four operations $f^*, f_*, f^!, f_!$, geometric arguments and the vanishing [Tub23, Lemma 2.6] works with integral coefficients because Nori proved it in this generality in [Nor02, Proposition 2.2]), we know that the constant object Λ_X is admissible on X. To be more precise, we first prove by induction on n the admissibility of the constant object on $\mathbb{A}^n_{\mathbb{Q}}$ (the case $n = 0$ is true because we already know the theorem in that case by Theorem 2.3.6, while the induction uses well-chosen projection to $\mathbb{A}^{n-1}$ and the already cited [Tub23, Lemma 2.6]). Then, using Noether normalisation one can prove that the result is true for any affine scheme. Finally, because our scheme is separated, we can do an induction on the numbers of affine open subsets needed to cover X and deduce that Λ_X is indeed admissible on X

The next step is to prove that any torsion-free lisse sheaf L on X is admissible. This follows from the fact that because L is dualisable, the functor $-\otimes L$ is t-exact because L is torsion-free (this can be checked after realisation and then on stalks so that this reduces to the same statement about modules) and left adjoint of a t-exact functor; thus by [Tub23, Corollary 2.13] the object $\Lambda \otimes L = L$ is admissible. Using [Tub23, Corollary 2.13] again, the functor $f_!$ for f étale preserves admissibility: the functors f^* and $f_!$ are t-exact because they are t-exact after taking the Betti realisation. Moreover by [Tub23, Lemma 2.10], if one has a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

with M' and M'' admissible, then M is admissible. Now if L is a lisse motive, then we have the short exact sequence

$$0 \to L_{\mathrm{tors}} \to L \to L_{\mathrm{fr}} \to 0$$

with L_{tors} the biggest torsion subobject and L_{fr} torsion-free. Note that L_{tors} and L_{fr} are also lisse because they are after taking the Betti realisation. Hence L_{fr} is admissible.

We claim that L_{tors} is also admissible. As it is torsion, it is in fact of the form $\iota(F)$ for a lisse étale sheaf on X: it is of the form $\iota(F)$ because of Proposition 3.4.1 and F is lisse because the functor $\mathrm{D}(X_{\text{ét}},\Lambda) \to \mathrm{D}(X^{\mathrm{an}},\Lambda)$ is conservative and the image of F in $\mathrm{DN}(X,\Lambda)$ is dualisable, and thus its image in $\mathrm{D}(X^{\mathrm{an}},\Lambda)$ is also dualisable which yields the dualisability of F by conservativity. Choose some $f\colon U \to X$ étale such that $f^*F = N$ is a constant Λ-module. We can choose a surjection $\Lambda^n \to N$, so that by t-exactness of $f_!$, we have a composition of surjections $f_!\Lambda^n \twoheadrightarrow f_!f^*F \twoheadrightarrow F$.

Let K be the kernel of this composed surjection. It is also a lisse and torsion-free étale sheaf: the ring Λ is flat over $\mathbb{Z}$ which is a Dedeking domain, so that a subobject of a torsion-free object is again torsion-free. Thus, in $\mathrm{D}^b(\mathcal{M}_{\mathrm{ord}}(X,\Lambda))$ we have an exact triangle

$$\iota(K) \to \iota(f_!\Lambda^n) \to L_{\mathrm{tors}}.$$

Because $\iota(K)$ and $\iota(f_!\Lambda^n)$ are admissible, for any motive $P \in \mathcal{M}_{\mathrm{ord}}(X,\Lambda)$ the maps

$$\mathrm{map}_{D^b(\mathcal{M}(X,\Lambda))}(\iota(K),P) \to \mathrm{map}_{\mathrm{DN_{gm}}(X,\Lambda)}(\iota(K),P)$$

and

$$\mathrm{map}_{D^b(\mathcal{M}(X,\Lambda))}(\iota(f_!\Lambda^n),P) \to \mathrm{map}_{\mathrm{DN_{gm}}(X,\Lambda)}(\iota(f_!\Lambda^n),P)$$

are equivalences. Thus the analogous map for L_{tors} is an equivalence so that L_{tors} is also admissible. This implies that L is admissible. Now, the same proof as [Tub23, Theorem 2.18] (which is the same proof as [Nor02, Proposition 3.10]) gives by dévissage that any motive is admissible, finishing the proof. ☐

4.4 Applications: motivic Leray spectral sequence; arc descent

The existence of ordinary integral Nori motives implies that the Leray spectral sequence, with integral coefficients, is motivic; this is a generalisation of [IM24, Section 5.4] and [Ara05].

Proposition 4.4.1. *Let X be a finite type k-scheme with k a subfield of $\mathbb{C}$. Let $\mathcal{F}$ be an object of the abelian category $\mathrm{Sh_{cons}}(X^{\mathrm{an}},\mathbb{Z})$ of constructible sheaves of abelian groups over X. Assume that $\mathcal{F}$ is motivic, meaning that it is the image of some integral Nori motive under the Betti realisation. Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be morphisms of finite type $\mathbb{C}$-schemes. Then the Leray spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{R}^p g_* \mathrm{R}^q f_* \mathcal{F} \Rightarrow \mathrm{R}^{p+q}(g \circ f)_* \mathcal{F}$$

is the image of a spectral sequence in $\mathcal{M}_{\mathrm{ord}}(Z)$.

Proof. This is a consequence of the existence of the t-structure and the functoriality of pushforwards. ☐

As an application of the existence of the ordinary t-structure, we prove that Nori motives afford arc-descent.

Theorem 4.4.2. *The functor $\mathrm{DN_{gm}}(-,\mathbb{Z})$ satisfies arc-hyperdescent over finite-dimensional qcqs $\mathbb{Q}$-schemes.*

Proof. As

$$\mathrm{DN_{gm}}(X,\mathbb{Z}) = \bigcup_n \mathrm{DN_{gm}}(X,\mathbb{Z})^{[-n,n]}$$

we see that to prove that $\mathrm{DN_{gm}}(-,\mathbb{Z})$ is an arc-hypersheaf, it suffices to show that $\mathrm{DN_{gm}}(-,\mathbb{Z})^{[-n,n]}$ is an arc-hypersheaf (hence even an arc-sheaf), where the truncation is taken for the ordinary t-structure.

The presheaf $\mathrm{DN_{gm}}(-,\mathbb{Z})^{[-n,n]}$ takes values in Cat_{2n+1} which is compactly generated by cotruncated objects (see [BM21, Example 3.6 (3)]). We consider

$$\mathcal{F} \colon (\mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs}})^{\mathrm{op}} \to \mathrm{Cat}_{2n+1}$$

its left Kan extension to $(\mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs}})^{\mathrm{op}}$ the category of non necessarily finite-dimensional qcqs $\mathbb{Q}$-schemes. By Theorem 3.2.2 we have that $\mathcal{F}(X) = \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})^{[-n,n]}$ for X of finite dimension. We may now apply the main theorem of [BM21]: a finitary v-sheaf taking values in an ∞-category which is compactly generated by cotruncated objects is an arc-sheaf if and only if it is excisive. By definition, $\mathcal{F}$ is finitary.

First, we want to prove that $\mathcal{F}$ is a v-sheaf. Given a v-covering $f\colon Y \to X$, we can write it as $\lim_i (f_i\colon Y_i \to X_i)$ with X_i and Y_i of finite type over $\mathbb{Q}$. Moreover, as the sheaf condition for f and $\mathcal{F}$ is finite because it is a totalisation in an ∞-category which is compactly generated by cotruncated objects, it is compatible with the filtered colimit of the continuity. Hence, it suffices to show that $\mathrm{DN}_{\mathrm{gm}}(-, \mathbb{Z})^{[-n,n]}$ is a v-sheaf when restricted to finite type $\mathbb{Q}$-schemes, but for Noetherian schemes the v-topology and the h-topology agree, so that this is follows from Proposition 3.1.5 and Theorem 3.2.2 (in fact in our setting there is a simpler proof thanks to the t-structure: one can do as in [BM21, Proposition 5.11], in which they reduce v-descent to descent along proper surjective map thanks to étale hyperdescent, and then the proper base change implies the result, with a combination of Barr-Beck-Lurie theorem and the finitary property. This proof essentially dates back to Deligne in [Del90]).

Next, we prove that $\mathcal{F}$ satisfies excision. Because any Milnor square can be approximated by Milnor squares of finite-dimensional schemes ([EHIK21a, Lemma 3.2.6]), it suffices to prove that $\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})^{[-n,n]}$ satisfies excision on finite type $\mathbb{Q}$-schemes. We first prove that this is true for the whole bounded category $\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$. The proof is entirely analogous to the proof of Milnor excision for $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}$ in [RT24, Proposition 5.3]: the proof is cut in two, the rationalised part which follows from [EHIK21b], and the torsion part that follows from [BM21]. To prove that $\mathrm{DN}_{\mathrm{gm}}(-, \mathbb{Z})^{[-n,n]}$ satisfies excision as well, it suffices to show that given a Milnor square

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "g"] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd},$$

if M belongs to $\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$ and is such that f^*M and i^*M are in cohomological degrees $[-n, n]$ then in fact M is also in degrees $[-n, n]$. This is true because the map $Z \cup Y \to X$ is surjective hence the hypothesis imply that for any point x of X, the motive x^*M in $\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$ is concentrated in degrees $[-n, n]$, so that M is also concentrated in degrees $[-n, n]$. □

Remark 4.4.3. Conjecturally, the category $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbb{Z})$ has an ordinary t-structure after rationalisation. In that case the functor $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbb{Z}) \to \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$ is an equivalence because it is an equivalence rationalised and mod n for any nonzero integer n, so that $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(-, \mathbb{Z})$ is a finitary arc-hypersheaf and the category $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(X, \mathbb{Z})$ has an ordinary t-structure without needing to rationalise. In fact, it would suffice to prove that $\mathrm{DM}_{\mathrm{gm}}^{\text{ét}}(\mathrm{Spec}(\mathbb{Q}), \mathbb{Q})$ has a motivic t-structure.

5 Perverse Nori motives.

5.1 The perverse t-structure

In this subsection, we define the perverse t-structure on Nori motives. The definition is very similar to that of the perverse t-structure on constructible complexes of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules given in [BBD82, Section 2.2]. We also prove that the derived category of perverse Nori motives is $\mathrm{DN}_{\mathrm{gm}}$, which is in the spirit of [Bei87].

Proposition 5.1.1. *Let X be an excellent finite-dimensional $\mathbb{Q}$-scheme endowed with a dimension function δ*[2]*. We let*

$$^{p}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\leqslant 0} := \{M \in \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \mid \forall x \in X, i_x^*(M) \leqslant -\delta(x)\},$$

$$^{p}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\geqslant 0} := \{M \in \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \mid \forall x \in X, i_x^!(M) \geqslant -\delta(x)\}.$$

Then, the pair $({}^{p}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\leqslant 0}, {}^{p}\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})^{\geqslant 0})$ defines a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ such that the ℓ-adic realisation functor

$$R_\ell \colon \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) \to \mathrm{D}_{\mathrm{cons}}(X_{\mathrm{pro\acute{e}t}},\mathbb{Z}_\ell)$$

is t-exact when the right-hand side is endowed with its perverse t-structure.

Proof. The proof is the same as in [BBD82, Section 2.2] but we include a sketch for the reader's convenience. Consider couples $(\mathcal{S},\mathcal{L})$ where $\mathcal{S}$ is a finite stratification of X by locally closed connected regular subschemes, and $\mathcal{L}$ is the data, for each stratum Z of $\mathcal{S}$ of a finite set $\mathcal{L}(Z)$ of dualisable Nori motives on Z. We denote by $\mathrm{DN}_{(\mathcal{S},\mathcal{L})}(X,\mathbb{Z})$ the full subcategory of $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ made of $(\mathcal{S},\mathcal{L})$-constructible Nori motives, that is the motives M such that, for each stratum Z of $\mathcal{S}$ and each integer n, $\mathrm{H}^n(M|_Z)$ is isomorphic to an iterated extension of elements of $\mathcal{L}(Z)$.

Any such couple can be refined into a couple $(\mathcal{S},\mathcal{L})$ such that if Z is a stratum in $\mathcal{S}$ and N belongs to $\mathcal{L}(Z)$, letting $j\colon Z \to X$ be the inclusion, the Nori motive $j_*(N)$ is $(\mathcal{S},\mathcal{L})$-constructible.

This assumption ensures that we can define a t-structure on $\mathrm{DN}_{(\mathcal{S},\mathcal{L})}(X,\mathbb{Z})$ by gluing (in the sense of[BBD82, Section 1.4])the t-structure on each stratum Z by $-\delta(Z)$. If $(\mathcal{S},\mathcal{L})$ refines $(\mathcal{S}',\mathcal{L}')$, the t-structure on $\mathrm{DN}_{(\mathcal{S},\mathcal{L})}(X,\mathbb{Z})$ induces the t-structure on $\mathrm{DN}_{(\mathcal{S}',\mathcal{L}')}(X,\mathbb{Z})$; this is a consequence of the absolute purity property for Nori motives:

Lemma 5.1.2. *Let $i\colon Z \to X$ be a closed immersion everywhere of codimension c between regular schemes and let M be a dualisable object of $\mathrm{DN}(Z,\mathbb{Z})$, then the natural map*

$$i^*(M)(-c)[-2c] \to i^!(M)$$

is an equivalence.

Proof. This follows from the same statement for ℓ-adic sheaves [ILO14, Exposé XVI, Théorème 3.1.1] and from the conservativity of the ℓ-adic realisation Proposition 3.4.5 and its compatibility with the six functors over quasi-excellent schemes Construction 3.4.4. ☐

To finish the proof of the proposition, note that

$$\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z}) = \mathrm{colim}_{(\mathcal{S},\mathcal{L})}\ \mathrm{DN}_{(\mathcal{S},\mathcal{L})}(X,\mathbb{Z})$$

and therefore we get a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ by Lemma 1.2.2. The t-exactness of the realisation follows from the fact that the perverse t-structure is defined in the same way by gluing along stratifications in [BBD82, Section 2.2]. The characterisation of the positive and negative parts of the t-structure then follows from the t-exactness of the realisation combined with its conservativity and the analogous characterisation of the positive and negative parts of the perverse t-structure on $\mathrm{D}_{\mathrm{cons}}(X_{\mathrm{pro\acute{e}t}},\mathbb{Z}_\ell)$ which is [BBD82, 2.2.12]. ☐

---

[2] See [ILO14, Exposé XIV, Définition 2.1.8] for the definition of a dimension function.

Remark 5.1.3. Along the way we proved that the perverse t-structure on the subcategory $\mathrm{DN}_{\mathrm{lisse}}(X, \mathbb{Z})$ of $\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$ made of dualisable objects is the ordinary t-structure shifted by $-\delta(X)$.

This result also implies that any t-exactness property of the perverse t-structure on ℓ-adic sheaves is true for the perverse t-structure on Nori motives. In particular, the Lefschetz Hyperplane Theorem (also known as Artin Vanishing) is true for Nori motives. The Hard Lefschetz theorem for Nori motives is also true using similar arguments.

Definition 5.1.4. Let X be an excellent finite-dimensional $\mathbb{Q}$-scheme endowed with a dimension function. The *abelian category of perverse Nori motives* is the heart of the perverse t-structure on $\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$. We denote it by $\mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})$.

Remark 5.1.5. One of the main feats of the perverse t-structure with rational coefficients is that it is preserved by Verdier duality. This is famously not the case with integral coefficients (note for instance that $\mathrm{map}_{\mathrm{D}(\mathbb{Z})}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}[-1]$). However, if ℓ is a prime number there is a t-structure on $\mathrm{D}_{\mathrm{cons}}(X_{\text{proét}}, \mathbb{Z}_\ell)$ that is exchanged with the perverse t-structure by duality (see [BBD82, 4.0 (a)]).

We can mimic its definition to get a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$ as follows; assume that $\mathbb{Z}$ is a Dedekind domain and define

$$^{\mathrm{ord}^+}\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})^{\leqslant 0} := \{M \in \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z}) \mid M \leqslant_{\mathrm{ord}} 1 \text{ and } \mathrm{H}^1(M) \otimes_{\mathbb{Z}} \mathbb{Q} = 0\}$$

$$^{\mathrm{ord}^+}\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})^{\geqslant 0} := \{M \in \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z}) \mid M \geqslant_{\mathrm{ord}} 0 \text{ and } \mathrm{H}^0(M) \text{ is torsion-free}\}$$

This is a t-structure on $\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})$; then by gluing as before this t-structure along stratifications (using the same shifts) we get a t-structure $({}^{p^+}\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})^{\leqslant 0}, {}^{p^+}\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})^{\geqslant 0})$ whose heart we denote by $\mathcal{M}^+_{\mathrm{perv}}(X, \mathbb{Z})$.

Because the six functors are compatible with the ℓ-adic realisation and because same statement is true after realisation by [BBD82, 3.3.4], the Verdier duality functor $\mathbb{D}_X$ of Proposition 3.3.3 exchanges the subcategory ${}^{p^+}\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})^{\leqslant 0}$ (resp. ${}^{p^+}\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})^{\geqslant 0}$) with ${}^{p}\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})^{\geqslant 0}$ (resp. ${}^{p^+}\mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})^{\leqslant 0}$) and therefore it exchanges $\mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})$ with $\mathcal{M}^+_{\mathrm{perv}}(X, \mathbb{Z})$.

5.2 Universal motives and the derived category of perverse Nori motives

By the universal property of the bounded derived category, there is a canonical realisation functor

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})) \to \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z}). \tag{5.2.0.1}$$

This functor is an equivalence after tensorisation by $\mathrm{Perf}_{\mathbb{Q}}$. We have the following rigidity result to compute its torsion:

Proposition 5.2.1. *Let X be an excellent finite-dimensional $\mathbb{Q}$-scheme endowed with a dimension function δ. Let n be a prime number. Consider the composition*

$$\iota_n \colon \mathrm{Perv}(X_{\text{ét}}, \mathbb{Z}/n) \subseteq \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \mathbb{Z}/n) \xrightarrow{f} \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \mathbb{Z})_{tors} \xrightarrow{\iota} \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z})_{tors}$$

where $\mathrm{Perv}(X_{\text{ét}}, \mathbb{Z}/n)$ *is the category of étale perverse sheaves and f is the forgetful functor. It lands in $\mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})[n]$ and induces an equivalence*

$$\mathrm{Perv}(X_{\text{ét}}, \mathbb{Z}/n) \xrightarrow{\sim} \mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})[n].$$

Proof. By Proposition 3.4.1 the last functor is an equivalence. Moreover as $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})_{tors}$ is the kernel of the perverse t-exact functor $-\otimes_{\mathbb{Z}}\mathbb{Q}$, it is endowed with a perverse t-structure, whose heart is the abelian category of torsion perverse motives.

First note that the forgetful functor f is t-exact: indeed, the heart of the perverse t-structure on $\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}},\mathbb{Z})$ consists of those complexes K such there is a stratification of X by nil-regular subschemes such that on each stratum S we have $K|_S \simeq L[-\delta(X)]$ for a lisse sheaf L; the image through the forgetful functor of a lisse sheaf of $\mathbb{Z}/n$-modules is a lisse sheaf of $\mathbb{Z}$-modules, whence the functor f is indeed t-exact.

This implies that the functor ι_n lands in $\mathcal{M}_{\mathrm{perv}}(\mathbb{Z})[n]$. Furthermore, the restriction of the forgetful functor to the heart affords a left adjoint, namely ${}^p\mathrm{H}^0(-\otimes_{\mathbb{Z}}\mathbb{Z}/n)$; it is in fact the underived tensor product with $\mathbb{Z}/n$ that sends a perverse sheaf K to the cokernel of the map $K \xrightarrow{\times n} K$. With this, it is clear that if M belongs to $\mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n)$, we have

$$\mathrm{H}^0(f(M)\otimes_{\mathbb{Z}}\mathbb{Z}/n) = M$$

and for any object of $\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}},\mathbb{Z})^{\heartsuit}_{tors}$ which is of n-torsion, we have

$$f(\mathrm{H}^0(N\otimes_{\mathbb{Z}}\mathbb{Z}/n)) = N.$$

Whence, the functor ι_n indeed induces an equivalence. □

Unfortunately, we do not know if the abelian category $\mathcal{M}_{\mathrm{perv}}$ has enough torsion-free sheaves. If it was true, we would use Proposition 1.2.5 to conclude that the functor (5.2.0.1) is fully faithful thanks to Proposition 5.2.1 and Beilinson's Theorem on the derived category of perverse sheaves [Bei87]; this would then imply that this functor is an equivalence. We have a partial answer in that direction, that requires to introduce the universal category of perverse motives. Let k be a subfield of the complex numbers (a more general field can be considered, but the following results for those fields can be reduced to subfields of $\mathbb{C}$ by the continuity properties of perverse categories considered along fields).

Recall first the theory of universal abelian categories in the form used in [IM24, Section 2.1]:

Construction 5.2.2. Given a triangulated category T, one first consider the category of finite presentation modules over T: it is the full subcategory $\mathrm{fp}(\mathrm{T}) \subset \mathrm{Sh}_{\mathrm{add}}(\mathrm{T},\mathrm{Ab})$ of additive functors $\mathcal{F}$ which fit in a short exact sequence

$$y_M \to y_N \to \mathcal{F} \to 0$$

where y_M is the image of M under the Yoneda embedding. Any cohomological functor on T will factor through $\mathrm{fp}(\mathrm{T})$. Thus, starting with a cohomological functor

$$\mathrm{H}\colon \mathrm{T} \to \mathcal{A}$$

with values in an abelian category $\mathcal{A}$, we have a factorisation of H of the form

$$\mathrm{T} \to \mathrm{fp}(\mathrm{T}) \xrightarrow{\bar{\mathrm{H}}} \mathcal{A}.$$

Denote by I the kernel of $\bar{\mathrm{H}}$. It is the full subcategory of $\mathrm{fp}(\mathrm{T})$ consisting of objects M such that $\bar{\mathrm{H}}(M) = 0$. The universal factorisation of H is the factorisation

$$\mathrm{T} \to \mathcal{M}(\mathrm{T},\mathrm{H}) := \mathrm{fp}(\mathrm{T})/I \xrightarrow{\mathrm{H}_{\mathrm{univ}}} \mathcal{A}.$$

Is has clearly an universal property, and note that composing H with any faithful functor will give the same universal category $\mathcal{M}(\mathrm{T},\mathrm{H})$. And that any functor comparing with H after composing with an exact functor $F:\mathcal{A}\to\mathcal{B}$ will also factor uniquely through $\mathcal{M}(\mathrm{T},\mathrm{H})$ (because one kernel will be included in the other). For example, this is the case for $\mathcal{A}=\Lambda\text{-mod}\xrightarrow{\otimes_\Lambda\Lambda'}\Lambda'\text{-mod}=\mathcal{B}$ with Λ' a flat Λ-algebra.

We apply this to the perverse realisation functor.

Construction 5.2.3. Let X be a finite type k-scheme. Consider the cohomological functor

$$ {}^p\mathrm{H}^0\circ\rho_\mathrm{B}\colon \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Z})\to\mathrm{Perv}(X^{\mathrm{an}},\mathbb{Z}) $$

where $\mathrm{Perv}(X^{\mathrm{an}},\mathbb{Z})$ is the category of analytic perverse sheaves and ${}^p\mathrm{H}^0$ is the 0-th perverse cohomology functor. As in Construction 5.2.2, we can consider the universal abelian factorisation $\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})$ of this functor through a faithful exact functor $r_\mathrm{B}\colon\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})\to\mathrm{Perv}(X^{\mathrm{an}},\mathbb{Z})$. We thus have a universal homological functor

$$ \mathrm{H}_{\mathrm{univ}}\colon \mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Z})\to\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z}). $$

Because the Betti realisation of perverse Nori motives is faithful exact (as the realisation from $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ is conservative), we actually have a finer factorisation:

$$\begin{tikzcd}
\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Z}) \arrow[r, "\mathrm{H}_{\mathrm{univ}}"] \arrow[rr, bend right, "{}^p\mathrm{H}^0\circ\rho_\mathrm{N}"'] & \mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z}) \arrow[r, "\mathrm{comp}"] & \mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})
\end{tikzcd}$$

with comp a faithful exact functor.

Lemma 5.2.4. *Let X be a finite type k-scheme. The functor*

$$ \mathrm{comp}\otimes_\mathbb{Z}\mathbb{Q}\colon\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})\otimes_\mathbb{Z}\mathbb{Q}\to\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})\otimes_\mathbb{Z}\mathbb{Q} $$

is an equivalence.

Proof. The definition of $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Q})$ is $\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Q})$. Thus we have to show that $\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})\otimes_\mathbb{Z}\mathbb{Q}$ is the universal category for ${}^p\mathrm{H}^0\circ\rho_\mathrm{B}\otimes_\mathbb{Z}\mathbb{Q}$. This follows easily from the construction of the universal category together with the fact that the rationalisation of the abelian hull of an additive category if the abelian hull of the rationalisation. ☐

For each nonzero integer n, we have an Artin motive functor

$$ \iota\colon\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}},\mathbb{Z}/n)\to\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X,\mathbb{Z}) $$

such that the composition with the realisation is perverse t-exact and conservative. Thus it induces a faithful exact functor

$$ i\colon\mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n)\to\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})[n]. $$

The following result is an adaptation of a proof due to Nori ([Fak00]) in the case of fields.

Proposition 5.2.5. *Let X be a finite type k-scheme. Let n be an integer. The functor*

$$ i\colon\mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n)\to\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})[n] $$

is an equivalence of quasi-inverse the functor

$$ \mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})[n]\xrightarrow{\mathrm{comp}}\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})[n]\simeq\mathrm{Perv}(X_{\text{ét}},\mathbb{Z}/n). $$

Proof. We begin by noting three things:

1. The composite functor $\mathrm{comp} \circ i$ lands in $\mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})[n] = \mathrm{Perv}(X_{\text{ét}}, \mathbb{Z}/n)$ and induces the identity of the latter, because we have that $\mathrm{comp} \circ i = {}^p\mathrm{H}^0 \circ \rho_{\mathrm{N}} \circ \iota$.
2. The functors comp and i are faithful; thus it suffices to prove that i is essentially surjective. Indeed, if i is essentially surjective, this implies that comp is fully faithful, thus i is also fully faithful.
3. The image of i is stable under subquotients. Indeed, if M belongs to $\mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})[n]$ and is a subobject of something of the form $i(N)$, then $\mathrm{comp}(M)$ is a subobject of $\mathrm{comp} \circ i(N) = N$. This implies that $i \circ \mathrm{comp}(M)$ and M are both subobjects of $i(N)$ that are isomorphic after applying the faithful functor comp thus they are isomorphic. For the quotients, one uses that if there is a surjection $i(N) \to M$ then the kernel is in the image of i thus the cokernel also.

Note also that if N belongs to $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbb{Z})$, we can write $N/n = \iota(K)$ for some K in $\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \mathbb{Z}/n)$, thus

$$\mathrm{H}_{\mathrm{univ}}(N/n) = \mathrm{H}_{\mathrm{univ}}(\iota(K)) \overset{(*)}{=} \mathrm{H}_{\mathrm{univ}}(\iota({}^p\mathrm{H}^0(K))) = i({}^p\mathrm{H}^0(K))$$

is in the image of i, where $(*)$ follows from the perverse t-exactness of the functor

$$\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \mathbb{Z}/n) \to \mathrm{DN}_{\mathrm{gm}}(X, \mathbb{Z}).$$

It suffices to show that any object of the form M/nM is in the image of i, because any n-torsion object is of this form. Thus, let M be in $\mathcal{M}_{\mathrm{univ}}(X, \mathbb{Z})$ and let $\mathrm{H}_{\mathrm{univ}}(N) \to M$ be a surjective map (this exists by definition of the universal category see Construction 5.2.2). As there is an exact triangle

$$N \xrightarrow{\times n} N \to N/n$$

in $\mathrm{DM}^{\text{ét}}_{\mathrm{gm}}(X, \mathbb{Z})$, we have a short exact sequence

$$\mathrm{H}_{\mathrm{univ}}(N) \xrightarrow{\times n} \mathrm{H}_{\mathrm{univ}}(N) \to \mathrm{H}_{\mathrm{univ}}(N/n).$$

Denote by t the image of the map $\mathrm{H}_{\mathrm{univ}}(N) \to \mathrm{H}_{\mathrm{univ}}(N/n)$. Because $\mathrm{H}_{\mathrm{univ}}(N/n)$ is in the image of i we see that t is also in the image of i as the latter is closed under subquotients. Moreover we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{H}_{\mathrm{univ}}(N) \arrow[r, "\times n"] \arrow[d] & \mathrm{H}_{\mathrm{univ}}(N) \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, dashed] & 0 \\
M \arrow[r, "\times n"'] & M \arrow[r] & M/nM \arrow[r] & 0
\end{tikzcd}$$

where the map $T \to M/nM$ exists because the composite of the map $M \to M/nM$ with the map $\mathrm{H}_{\mathrm{univ}}(N) \to M$ vanishes on the kernel of the quotient map $\mathrm{H}_{\mathrm{univ}}(N) \to T$. As $\mathrm{H}_{\mathrm{univ}}(N) \to M$ is surjective, this is also the case of the map $T \to M/nM$ thus M/nM is in the image, finishing the proof. ☐

Thus the functor

$$\mathrm{comp}\colon \mathcal{M}_{\mathrm{univ}}(X, \mathbb{Z}) \to \mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})$$

is an equivalence after being tensored by $\mathbb{Q}$ and induces an equivalence on n-torsion objects for all n. To conclude that it is an equivalence, one could compute some Ext^1 in $\mathcal{M}_{\mathrm{perv}}(X, \mathbb{Z})$, but this seems to be beyond the scope of our current technology. However, the following conjecture seems more reachable:

Conjecture 5.2.6. *Let X be an algebraic variety over $\mathbb{Q}$. Then the abelian category $\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})$ has enough torsion-free object: for every M in $\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})$ there exists an epimorphism $N \twoheadrightarrow M$ in $\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})$ with N a torsion-free object.*

This conjecture would follow from the equivalence

$$\mathcal{M}_{\mathrm{pairs}}(X,\mathbb{Z}) \xrightarrow{\sim} \mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z}),$$

where $\mathcal{M}_{\mathrm{pairs}}(X,\mathbb{Z})$ is the universal category associated to a diagram of very good pairs as in Nori's original construction, as considered by F. Ivorra in [Ivo17] (he was considering rational motives but his construction remains valid for integral sheaves). This equivalence would follow from the fact that Ivorra's realisation functor, the main construction in [Ivo16] is the ${}^p\mathrm{H}^0$ of our functor ρ_{N}. Note that in [Ivo16] the author was only considering smooth varieties but it is not hard to see that a slight modification of his functor (namely, replace the constant sheaf $\mathbb{Q}[-\dim X]$ on X by $\pi_X^!\mathbb{Q}_{\mathrm{Spec}(\mathbb{Q})}$) would give a functor for any variety. Such a conjecture is related to [IM24, Conjecture 2.12], and might be doable with the current technology.

We now explain how Conjecture 5.2.6 implies that both $\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z}) \simeq \mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$ for all X but also that $\mathrm{D}^b(\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})) \simeq \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$.

Proposition 5.2.7. *Assume that Conjecture 5.2.6 holds. Then for every field k of characteristic zero and X a finite type k-scheme, the functor* $\mathrm{comp}\colon \mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z}) \to \mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$ *is an equivalence, and the functor*

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})) \to \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$$

is an equivalence.

Proof. By continuity along change of fields, it suffices to prove the theorem when X is of finite type over $\mathbb{Q}$. It suffices to prove that the functor

$$\mathrm{D}^b(\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})) \to \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$$

induced by universal property from the functor

$$\mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z}) \to \mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z}) \to \mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$$

is fully faithful. Indeed, once it is fully faithful, it will be an equivalence by dévissage because the essential image is then stable under extension. Indeed if $M \in \mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$, then there is an short exact sequence

$$0 \to M_{\mathrm{tors}} \to M \to M_{\mathrm{fr}} \to 0$$

in $\mathcal{M}_{\mathrm{perv}}(X,\mathbb{Z})$ with M_{tors} of torsion and M_{fr} torsion-free. By Proposition 5.2.5 and Proposition 5.2.1 the object M_{tors} is in the image. Moreover by Lemma 5.2.4 there exists $P \in \mathcal{M}_{\mathrm{univ}}(X,\mathbb{Z})$ without torsion and a map $P \to M_{\mathrm{fr}}$ which becomes an equivalence after tensorisation by $\mathbb{Q}$. In particular, this map has to be injective, and the cokernel is torsion, thus in the image. By fully faithfulness, the object M_{fr} is in the image, whence M is in the image.

The functor is an equivalence after tensorisation by $\mathbb{Q}$, and if we tensor the mapping spectra by $\otimes_{\mathbb{Z}}\mathbb{Z}/p\mathbb{Z}$, using Proposition 1.2.5 (we have enough torsion-free objects by hypothesis!) together with [Bei87] and rigidity, we conclude. Note that this also proves that comp is an equivalence. ☐

5.3 Applications to Artin motives

Recall that given a scheme X and a ring of coefficients $\mathbb{Z}$, the category of Artin motives (resp. constructible Artin motives) is the localising (resp. thick) subcategory of $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$ generated by the $f_*\mathbb{1}_Y$ for $f\colon Y\to X$ finite (see for instance [Rui22, Definition 2.1.1.1]); we denote it by $\mathrm{DM}^{\text{ét},0}(X,\mathbb{Z})$ (resp. $\mathrm{DM}^{\text{ét},0}_c(X,\mathbb{Z})$). We likewise define $\mathrm{DN}^0(X,\mathbb{Z})$ and $\mathrm{DN}^0_c(X,\mathbb{Z})$.

Proposition 5.3.1. *Let X be a qcqs $\mathbb{Q}$-scheme and let Λ be a commutative ring. Then, the Nori realisation functor ρ_N induces an equivalence $\mathrm{DM}^{\text{ét},0}(X,\Lambda)\to\mathrm{DN}^0(X,\Lambda)$. Moreover, the functor $\mathrm{DM}^{\text{ét},0}_c(X,\mathbb{Z})\to\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ is t-exact when $\mathrm{DM}^{\text{ét},0}_c(X,\mathbb{Z})$ is endowed with the ordinary homotopy t-structure (see [Rui22, Definition 3.1.1.1 and Theorem 3.1.2.7]).*

Proof. The functors $\mathrm{Mod}_\Lambda(\mathrm{DN}^0(X,\mathbb{Z}))\to\mathrm{DN}^0(X,\Lambda)$ and $\mathrm{Mod}_\Lambda(\mathrm{DM}^{\text{ét},0}(X,\mathbb{Z}))\to\mathrm{DM}^{\text{ét},0}(X,\Lambda)$ are equivalences. Hence, we can assume that $\Lambda=\mathbb{Z}$. Using Lemma 1.3.1, it suffices to tackle the case where $\Lambda=\mathbb{Z}/n\mathbb{Z}$ and the case where $\Lambda=\mathbb{Q}$. The first case follows from rigidity while the second case follows from the main Theorem of [Tub24a]. The t-exactness property follows from the t-exactness of the ℓ-adic realisations functors on Artin motives ([Rui22, Theorem 3.1.2.7]) and their conservativity and t-exactness on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$. □

We will now use Nori motives to give a characterisation of dualisable objects of $\mathrm{DM}^{\text{ét},0}_c(X,\mathbb{Z})$. To that end, recall that the category $\mathrm{DM}^{\text{ét},\mathrm{sm}0}(X,\mathbb{Z})$ (resp. $\mathrm{DM}^{\text{ét},\mathrm{sm}0}_c(X,\mathbb{Z})$) of smooth Artin motives (resp. constructible smooth Artin motives) is the localising (resp. thick) subcategory of $\mathrm{DM}^{\text{ét}}(X,\mathbb{Z})$ generated by the $f_*\mathbb{1}_Y$ for $f\colon Y\to X$ finite *étale*. Using Proposition 5.3.1, they are equivalent to full subcategories of $\mathrm{DN}(X,\mathbb{Z})$ defined similarly.

Lemma 5.3.2. *Let Λ be a commutative ring. For X a qcqs $\mathbb{Q}$-scheme denote by $\mathrm{DM}^{\text{ét},0}_{\mathrm{rig}}(X,\Lambda)$ the full subcategory of $\mathrm{DM}^{\text{ét},0}(X,\Lambda)$ made of those objects which are dualisable in $\mathrm{DM}^{\text{ét}}(X,\Lambda)$. The functor $\mathrm{DM}^{\text{ét},0}_{\mathrm{rig}}(-,\Lambda)$ is a finitary étale sheaf.*

Proof. As $\mathrm{DM}^{\text{ét}}_c(-,\Lambda)$ is finitary by [RT24, Corollary 3.10], the functor $\mathrm{DM}^{\text{ét},0}_{\mathrm{rig}}(-,\Lambda)$ is also finitary: being Artin and rigid are both finitary conditions. We have to show that $\mathrm{DM}^{\text{ét},0}_{\mathrm{rig}}(-,\Lambda)$ is a subsheaf of $\mathrm{DM}^{\text{ét}}_c(-,\Lambda)$, the latter being a sheaf by [RT24, Lemma 2.5], meaning that if $f\colon Y\to X$ is an étale covering and M is an object of $\mathrm{DM}^{\text{ét}}_c(X,\Lambda)$ such that f^*M is dualisable and Artin, then so is M. But dualisability is étale local and so is being a 0-motive. □

Proposition 5.3.3. *Let X be a normal $\mathbb{Q}$-scheme. The inclusion*

$$\mathrm{DM}^{\text{ét},\mathrm{sm}0}_c(X,\mathbb{Z})\subseteq\mathrm{DM}^{\text{ét},0}_{\mathrm{rig}}(X,\mathbb{Z})$$

is an equivalence. The t-structure on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ restricts to the full subcategory $\mathrm{DM}^{\text{ét},\mathrm{sm}0}_c(X,\mathbb{Z})$ and the heart of the restricted t-structure is equivalent to $\mathrm{Rep}^A(\pi_1^{\text{ét}}(X),\mathbb{Z})$.

Proof. Using étale descent, continuity and Lemma 5.3.2, it suffices to prove the statements if X is of finite type over an algebraically closed field.

First note that the inclusion $\mathrm{DM}^{\text{ét},0}_{\mathrm{rig}}(X,\mathbb{Z})\to\mathrm{DN}^0_{\mathrm{rig}}(X,\mathbb{Z})$ is an equivalence, where the right-hand side consists of the objects of $\mathrm{DN}^0(X,\mathbb{Z})$ which are dualisable in $\mathrm{DN}(X,\mathbb{Z})$. Indeed, if M belongs to $\mathrm{DN}^0_{\mathrm{rig}}(X,\mathbb{Z})$ then its dual is also a 0-motive, because being a constructible 0-motive can be tested on points and all constructible 0-motives over a point are dualisable, with 0-motivic duals. But now, the t-structure of $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ restricts to $\mathrm{DM}^{\text{ét},0}_{\mathrm{rig}}(X,\mathbb{Z})$ because it can be restricted to both dualisable objects and 0-motives.

By Proposition 4.1.3 we have a pullback square

$$\begin{tikzcd}
\mathrm{DM}^{\text{ét},0}_{\mathrm{rig}}(X,\mathbb{Z})^{\heartsuit} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Loc}(X^{\mathrm{an}},\mathbb{Z}) \arrow[d] \\
\mathrm{DM}^{\text{ét},0}_{\mathrm{rig}}(X,\mathbb{Q})^{\heartsuit} \arrow[r] & \mathrm{Loc}(X^{\mathrm{an}},\mathbb{Q})
\end{tikzcd}$$

where $\mathrm{Loc}(X^{\mathrm{an}},\mathbb{Z})$ is the abelian category of local systems on X^{an} with $\mathbb{Z}$ coefficients. By [Tub24a] the category $\mathrm{DM}^{\text{ét},0}_{\mathrm{rig}}(X,\mathbb{Q})^{\heartsuit}$ is equivalent (through $\rho_!$) to $\mathrm{Rep}^A(\pi_1^{\text{ét}}(X),\mathbb{Q})$. As the category $\mathrm{Rep}^A(\pi_1^{\text{ét}}(X),\mathbb{Z})$ also fits in the corner of this pullback diagram, this proves that the functor $\rho_!$ induces an equivalence $\mathrm{Rep}^A(\pi_1^{\text{ét}}(X),\mathbb{Z}) \to \mathrm{DM}^{\text{ét},0}_{\mathrm{rig}}(X,\mathbb{Z})^{\heartsuit}$. Finally, by dévissage, because any object of $\mathrm{Rep}^A(\pi_1^{\text{ét}}(X),\mathbb{Z})$ defines an object of $\mathrm{DM}^{\text{ét},\mathrm{sm}0}_c(X,\mathbb{Z})$, any rigid 0-motive is smooth Artin. ☐

Using the framework of Nori motives, we can prove new cases of [Rui22, Theorem 3.2.3.7] which is one of the main results of *loc. cit.* and is an analogue of the Artin Vanishing Theorem in the setting of Artin motives. We begin by the following lemma:

Lemma 5.3.4. *Let X be a qcqs $\mathbb{Q}$-scheme. The ordinary and perverse t-structures on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$ restrict to the category $\mathrm{DN}^{\mathrm{coh}}_c(X,\mathbb{Z})$ defined as the thick subcategory of $\mathrm{DN}(X,\mathbb{Z})$ generated by the $f_*\mathbb{Z}_Y$ where $f\colon Y\to X$ is proper.*

Proof. By [Tub24a], the result holds for $\mathbb{Q}$ coefficients. Moreover it suffices to prove the result over the spectrum of a field k. Because the map

$$\mathrm{DN}^{\mathrm{coh}}_c(k,\mathbb{Z})\otimes_{\mathrm{Perf}_{\mathbb{Z}}}\mathrm{Perf}_{\mathbb{Q}}\to \mathrm{DN}^{\mathrm{coh}}_c(k,\mathbb{Q})$$

is an equivalence and if M is in $\mathrm{DN}^{\mathrm{coh}}_c(k,\mathbb{Z})$ then $\mathrm{H}^i(M\otimes_{\mathbb{Z}}\mathbb{Q})\simeq \mathrm{H}^i(M)\otimes_{\mathbb{Z}}\mathbb{Q}$ is cohomological, there is a cohomological motive P in $\mathrm{DN}^{\mathrm{coh}}_c(k,\mathbb{Z})$ and a map $P\to \mathrm{H}^i(M)$ whose cone is torsion and dualisable, thus a constructible 0-motive, which are cohomological. The claim about the perverse t-structure is true by gluing. ☐

Recall that a smooth Artin motive M is $\mathbb{Q}$-constructible if $M\otimes_{\mathbb{Z}}\mathbb{Q}$ is constructible as a smooth Artin motive. We denote by $\mathrm{DM}^{\text{ét},sm0}_{\mathbb{Q}-c}(X,\mathbb{Z})$ the category of $\mathbb{Q}$-constructible smooth Artin motives. We can now prove new cases of [Rui22, Theorem 3.2.3.7].

Proposition 5.3.5. *Let S be an excellent scheme $\mathbb{Q}$-scheme, let $f\colon X\to S$ be an affine quasi-finite morphism. Assume that X is regular. Then, the functor*

$$f_!\colon \mathrm{DM}^{\text{ét},sm0}_{\mathbb{Q}-c}(X,\mathbb{Z})\to \mathrm{DM}^{\text{ét},0}(S,\mathbb{Z})$$

is t-exact with respect to the perverse homotopy t-structure (see [Rui22, Definition 3.2.1.1]).

Proof. The same proof as in [Rui22, Theorem 3.2.3.7] allows to reduce to the case of rational coefficients. We now have to prove that $f_!\colon \mathrm{DN}^{sm0}_c(X,\mathbb{Q})\to\mathrm{DN}^0(S,\mathbb{Q})$ is t-exact with respect to the perverse homotopy t-structure (defined in the same fashion as in [Rui22, Definition 3.2.1.1]).

The rest of the proof is an adaptation of the arguments of [Rui24, Section 4.8]. We can write the functor

$$f_!\colon \mathrm{DM}^{\text{ét},sm0}_c(X,\mathbb{Q})\to \mathrm{DM}^{\text{ét},0}(S,\mathbb{Q})$$

as the composition

$$\mathrm{DM}^{\text{ét},sm0}_c(X,\mathbb{Q})\simeq \mathrm{DN}^{sm0}_c(X,\mathbb{Q})\xrightarrow{\iota_0}\mathrm{DN}^{\mathrm{coh}}_c(X,\mathbb{Q})\xrightarrow{f_!}\mathrm{DN}^{\mathrm{coh}}_c(S,\mathbb{Q})\xrightarrow{\omega^0}\mathrm{DN}^0_c(S,\mathbb{Q})\simeq \mathrm{DM}^{\text{ét},0}_c(S,\mathbb{Q}),$$

where the Artin truncation functor

$$\omega^0 \colon \mathrm{DN}^{\mathrm{coh}}(X,\mathbb{Q}) \to \mathrm{DN}^0(X,\mathbb{Q})$$

defined as the right adjoint to the inclusion ι_0 of Artin motives into cohomological motives, lands in $\mathrm{DN}^0_c(X,\mathbb{Q})$ by [Tub24a]. Thus it suffices to check that all functors are t-exact. The functor ω^0 is t-exact with respect to the perverse t-structure on the left-hand side (which exists by the above Lemma 5.3.4) and the perverse homotopy t-structure on the right-hand side: this is because an object of $\mathrm{DN}^0(X,\mathbb{Q})$ is perverse homotopy t-negative if and only if it is perverse t-negative (this can be shown as in [Rui24, Proposition 4.5.1]) and because the functor ω^0 is perverse right t-exact as per [NV15, Lemma 3.2.6]. But as ι_0 is t-exact when restricted to smooth Artin objects because it is t-exact for the ordinary t-structure, the result reduces to the usual Artin Vanishing Theorem. □

5.4 Motivic intersection complex.

Consider an excellent scheme B of dimension 2 or less as well as a regular separated B-scheme of finite type S, endowed with a geometric and $\otimes$-invertible object ω_S in $\mathrm{DN}(S,\mathbb{Z})$ (note that $\omega_S = \mathbb{Z}_S$ works). We fix X a finite type S-scheme.

From now on, we assume that X is endowed with a bounded below dimension function δ; by shifting, we can assume 0 to be its lower bound. Any non-increasing function $\overline{q} \colon \mathbb{N} \to \mathbb{Z}$, defines a perversity function $q = \overline{q} \circ \delta$ on X along any stratification of X by connected regular and locally closed strata (in the sense of [BBD82, Section 2.1.1]). The construction given in Proposition 5.1.1 for $\overline{q} = -\mathrm{Id}$ works whenever the condition

$$(*)\colon \forall n \in \mathbb{N},\ q(n) - q(n+1) \in \{0,1,2\}$$

is satisfied. It then yields a q-perverse t-structure ${}^q\mathrm{t}$ on $\mathrm{DN}_{\mathrm{gm}}(X,\mathbb{Z})$, whose heart we will denote by $\mathcal{M}_{q-\mathrm{perv}}(X,\mathbb{Z})$ and whose cohomology functors are will be denoted by ${}^q\mathrm{H}^n(-)$.

Definition 5.4.1. Let $\overline{q} \colon \mathbb{N} \to \mathbb{Z}$ be a non-increasing function satisfying condition $(*)$, let $q = \overline{q} \circ \delta$ and let $j \colon U \to X$ be an open immersion. The *q-intermediate extension functor* is the functor

$$^{q}j_{!*} \colon \mathcal{M}_{q-\mathrm{perv}}(U,\mathbb{Z}) \to \mathcal{M}_{q-\mathrm{perv}}(X,\mathbb{Z})$$

defined by

$$^{q}j_{!*}M = \mathrm{Im}({}^q\mathrm{H}^0(j_!M) \to {}^q\mathrm{H}^0(j_*M)).$$

The *q-motivic intersection complex* is the q-perverse Nori motive

$$^{q}\mathrm{IC}_{X,\mathbb{Z}} := {}^{q}j_{!*}(\mathbb{Z}[-q(U)])[q(U)].$$

When X is endowed with a dimension function δ and $q = -\delta$ is the self-dual perversity, we simply refer to those as the intermediate extension functor $j_{!*}$ and the motivic intersection complex $\mathrm{IC}_{X,\mathbb{Z}}$.

Remark 5.4.2. In [Wil12, Wil19] Jörg Wildeshaus constructed a motivic intersection complex in $\mathrm{DM}(X,\mathbb{Q})$ using weights, in several cases (mainy when X is a Shimura variety). In those cases, we expect the complex $\mathrm{IC}_{X,\mathbb{Q}}$ to be the Nori realisation of Wildeshaus' motivic intersection complex up to a shift of $-\dim(X)$. Indeed assuming the existence of the motivic t-structure, this would be true by uniqueness of Wildeshaus' complex [Wil12, Theorem 3.1].

We fix a stratification (X_i) of X by regular and connected locally closed subschemes. Given an integer n, we set U_n to be the union of those strata X_i such that $\delta(X_i) \geqslant n$ and $j_n \colon U_n \to U_{n-1}$ to be the open immersion.

Proposition 5.4.3. *Let $\overline{q}\colon \mathbb{N} \to \mathbb{Z}$ be a non-increasing function satisfying condition $(*)$ and let $q = \overline{q} \circ \delta$. Let $k \geqslant 0$ be an integer and let $j\colon U_k \to X = U_0$ be the open immersion. Then there is a canonical isomorphism of functors*

$$^{q}j_{!*} \simeq \left(\tau^{\leqslant \overline{q}(0)-1}(j_1)_*\right) \circ \cdots \circ \left(\tau^{\leqslant \overline{q}(k-2)-1}(j_{k-1})_*\right) \circ \left(\tau^{\leqslant \overline{q}(k-1)-1}(j_k)_*\right)$$

on $\mathcal{M}_{q-\mathrm{perv}}(U_k)$, with $\tau^{\leqslant i}$ the truncations functors for the ordinary t-structure.

Proof. This is a consequence of [BBD82, Proposition 1.4.23] (see the proof of [BBD82, Proposition 2.1.11]). ☐

Corollary 5.4.4. *Let $\overline{q}, \overline{r}\colon \mathbb{N} \to \mathbb{Z}$ be non-increasing functions and $q = \overline{q} \circ \delta$ and $r = \overline{r} \circ \delta$. Let $k \geqslant 0$ be an integer and let $j\colon U_k \to X = U_0$ be the open immersion. There exists a canonical pairing in $\mathrm{DN}_c(X, \mathbb{Z})$*

$$^{q}j_{!*}(M[-q(U_k)]) \otimes {}^{r}j_{!*}(N[-r[U_k]]) \to {}^{q+r}j_{!*}(M \otimes N[-q(U_k) - r(U_k)])$$

for $M, N \in \mathcal{M}^{\mathrm{rig}}_{\mathrm{ord}}(U_k)$ motivic local systems (that is, objects in the ordinary heart which are dualisable).

Proof. This is a direct consequence of the formula for the intermediate extension given in Proposition 5.4.3, knowing that the functors of the form γ_* are lax-monoidal as right adjoints of monoidal functors and that for any M and N, we have a map

$$\tau^{\leqslant n}(M) \otimes \tau^{\leqslant m}(N) \to \tau^{\leqslant n+m}(M \otimes N)$$

given by the right t-exactness of the tensor product and adjunction. ☐

This yields a pairing

$$\mathrm{IC}^{q}_{X,\mathbb{Z}} \otimes \mathrm{IC}^{r}_{X,\mathbb{Z}} \to \mathrm{IC}^{q+r}_{X,\mathbb{Z}}$$

in $\mathrm{DN}(X, \mathbb{Z})$. Now consider the *top perversity* $\overline{t} = -2\mathrm{Id}_{\mathbb{Z}}$. Gluing on the stratification as in [GM83, Section 5.1], whenever $\overline{q} \leqslant \overline{t}$, there exists a map $\mathrm{IC}^{q}_{X,\mathbb{Z}} \to \omega_X[\delta(X)]$ where $\omega_X = \mathbb{D}_X(\mathbb{Z})$ is the dualising complex. Whence a map

$$\mathrm{IC}_{X,\mathbb{Z}} \otimes \mathrm{IC}_{X,\mathbb{Z}} \to \omega_X[\delta(X)]. \tag{5.4.4.1}$$

With rational caoefficients, this pairing induces an equivalence

$$\mathrm{IC}_{X,\mathbb{Q}} \xrightarrow{\sim} \mathbb{D}_X(\mathrm{IC}_{X,\mathbb{Q}})[-\delta(X)] \tag{5.4.4.2}$$

by [GM83, Section 5.3]. Let k be a field of characteristic zero.

Lemma 5.4.5. *Let $K \in \mathrm{DN}_c(k, \mathbb{Z})$ and $M \in \mathcal{M}(k, \mathbb{Z})$. For each $i \in \mathbb{Z}$ there exists an short exact sequence*

$$0 \to \mathrm{H}^1(\underline{\mathrm{Hom}}(\mathrm{H}^{-i+1}(K), M)) \to \mathrm{H}^i(\underline{\mathrm{Hom}}(K, M)) \to \mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{H}^{-i}(K), M)) \to 0$$

in $\mathcal{M}(k, \mathbb{Z})$.

Proof. This realises to the classical universal coefficient theorem of abelian groups (or $\mathbb{Z}_\ell$-modules if one takes ℓ-adic realisations). ☐

Definition 5.4.6. Assume that X is a finite type k-scheme. The intersection homology complex of X is the object

$$\mathrm{IH}(X,\mathbb{Z}) := p_*\mathrm{IC}_{X,\mathbb{Z}}$$

in $\mathrm{DN}_c(k,\mathbb{Z})$ where $p\colon X \to \mathrm{Spec}(k)$ is the structural morphism. Denote by $\mathrm{IH}_i(X,\mathbb{Z}) = \mathrm{H}^{-i}(\mathrm{IH}(X,\mathbb{Z}))$ the i-th intersection homology motive. There is also a version with compact support

$$\mathrm{IH}_c(X,\mathbb{Z}) := p_!\mathrm{IC}_{X,\mathbb{Z}}$$

and its homology motive $\mathrm{IH}_{i,c}(X,\mathbb{Z}) = \mathrm{H}^{-i}(\mathrm{IH}_c(X,\mathbb{Z}))$

With rational coefficients, we obtain from (5.4.4.2) a perfect pairing

$$\mathrm{IH}_{i,c}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{IH}_{d-i}(X,\mathbb{Q}) \to \mathbb{Q}_k(-d),$$

where d is the dimension of X.

We finish with a motivic lift of the results in [GS83, Section 3] providing a motivic Seifert linking pairing. The proof of the next proposition is the same as in *loc. cit.*

Proposition 5.4.7. *For $i \in \mathbb{Z}$ denote by $T_i(X)$ (*resp.* by $T_{i,c}(X)$) the maximal torsion subobject of $\mathrm{IH}_i(X,\mathbb{Z})$ (*resp.* of $\mathrm{IH}_{i,c}(X,\mathbb{Z})$). There is a pairing*

$$T_{i,c}(X) \times T_{d-i-1}(X) \to \mathbb{Q}/\mathbb{Z}(-d)$$

in $\mathcal{M}(k,\mathbb{Z})$.

Proof. We apply Lemma 5.4.5 and obtain a short exact sequence

$$0 \to \mathrm{H}^1(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i-1}(X,\mathbb{Z}),\mathbb{Z})) \to \mathrm{H}^{-i}(\underline{\mathrm{Hom}}(\mathrm{IH}(X,\mathbb{Z})[d](d),\mathbb{Z})) \to \mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i}(X,\mathbb{Z}),\mathbb{Z})) \to 0.$$

Using the exact triangle $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ and the fact that for any $M \in \mathcal{M}(k,\mathbb{Z})$ the object $\mathrm{H}^1(\underline{\mathrm{Hom}}(M,\mathbb{Q}))$ vanishes we obtain that

$$\mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i-1}(X,\mathbb{Z})(d),\mathbb{Q}/\mathbb{Z})) \xrightarrow{\sim} \mathrm{H}^1(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i-1}(X,\mathbb{Z})(d),\mathbb{Z}))$$

is an equivalence. Moreover because $\mathbb{Q}/\mathbb{Z}$ is torsion, the map

$$\mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{T}_{d-i-1}(X)(d),\mathbb{Q}/\mathbb{Z})) \to \mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i-1}(X,\mathbb{Z})(d),\mathbb{Q}/\mathbb{Z}))$$

is also an equivalence (this is true after realisation). Also any map $M \to \mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{IH}_{d-i}(X,\mathbb{Z})(d),\mathbb{Z}))$ with M torsion vanishes because it can be checked after realisation. Thus the composed map $T_{i,c}(X) \to \mathrm{IH}_{i,c}(X,\mathbb{Z}) \to \mathrm{H}^{-i}(\underline{\mathrm{Hom}}(\mathrm{IH}(X,\mathbb{Z})[d](d),\mathbb{Z}))$ obtained from applying $\mathrm{H}^{-i}p_!$ to (5.4.4.1) factors through the inclusion

$$\mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{T}_{d-i-1}(X)(d),\mathbb{Q}/\mathbb{Z})) \to \mathrm{H}^{-i}(\underline{\mathrm{Hom}}(\mathrm{IH}(X,\mathbb{Z})[d](d),\mathbb{Z})),$$

yielding

$$T_{i,c}(X) \to \mathrm{H}^0(\underline{\mathrm{Hom}}(\mathrm{T}_{d-i-1}(X)(d),\mathbb{Q}/\mathbb{Z})).$$

By adjunction (here we use that $\mathcal{M}(k,\mathbb{Z})$ has enough flat objects) this provides the desired pairing. □

Remark 5.4.8. The above pairing is non-degenerate when X is smooth. It is not non-degenerate in general, there is a topological obstruction introduced in [GS83] when X is singular and proper.

6 Integral mixed Hodge modules.

6.1 Definition of mixed Hodge modules with integral coefficients

Let X be a separated finite type $\mathbb{C}$-scheme. In [Sai90], M. Saito constructed an abelian category $\mathrm{MHM}(X)$ of mixed Hodge modules over X, which is a relative version of mixed Hodge structures, modelled on perverse sheaves, in the sense that if $X = \mathrm{Spec}(\mathbb{C})$, we have that $\mathrm{MHM}(X) \simeq \mathrm{MHS}^P_{\mathbb{Q}}(\mathbb{C})$ is the category of polarisable mixed Hodge structure over $\mathbb{C}$ with rational coefficients (which consists of mixed Hodge structure such that the graded pieces of the weight filtration are polarisable [Del71, Définition 2.1.15], it is a full subcategory of Deligne's mixed Hodge structures), and there is a faithful exact functor

$$\mathrm{rat}\colon \mathrm{MHM}(X) \to \mathrm{Perv}(X^{\mathrm{an}}, \mathbb{Q}).$$

Moreover, M. Saito constructed the six operations on the bounded derived category $\mathrm{D}^b(\mathrm{MHM}(-))$, in a way compatible with the functor rat. In [Tub23] the second author proved that the construction of these six operations by M. Saito could be canonically lifted to the world of ∞-categories, with all possible coherences (this includes an extension to non separated schemes and morphisms). As a consequence, using the universal property of $\mathrm{DM}^{\text{ét}}(-,\mathbb{Q})$, the second author obtained a realisation

$$\rho_{\mathrm{H}}\colon \mathrm{DM}^{\text{ét}}(-,\mathbb{Q}) \to \mathrm{D}_{\mathrm{H}}(-)$$

compatible with the six operations, where $\mathrm{D}_{\mathrm{H}}(-)$ is the indization of $\mathrm{D}^b(\mathrm{MHM}(-))$. By [Bei86] the abelian category of polarisable mixed Hodge structure with rational coefficients $\mathrm{MHS}^P_{\mathbb{Q}}(\mathbb{C})$ is of cohomological dimension 1. If X is a complex variety of dimension d with structural morphism π_X, as the cohomological amplitude of $(\pi_X)_*\underline{\mathrm{Hom}}(-,-)$ can be measured after applying the functor rat, it is at most $2d+1$; therefore the abelian category of $\mathrm{MHM}(X)$ is of cohomological dimension less that $2d+2$. Thus we have in fact that $\mathrm{D}_{\mathrm{H}}(X)$ is the unbounded derived category of ind-mixed Hodge modules $\mathrm{D}(\mathrm{Ind}\,\mathrm{MHM}(X))$ (because, for example by [CD16, Lemma 1.1.7] the functor $\mathrm{map}_{\mathrm{D}}(M,-)$ commutes with colimits when M an object of $\mathrm{MHM}(X)$).

In this section, we show that the methods we used for Nori motives apply for mixed Hodge modules, and that proofs are easier in this case.

Definition 6.1.1. The presentable ∞-category of mixed Hodge modules with integral coefficients is the pullback

$$\begin{tikzcd}
\mathrm{D}_{\mathrm{H}}(-,\mathbb{Z}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{D}_{\mathrm{H}}(-) \arrow[d, "\mathrm{rat}"] \\
\mathrm{Ind}\,\mathrm{D}_{\mathrm{cons}}((-)^{\mathrm{an}},\mathbb{Z}) \arrow[r, "-\otimes\mathbb{Q}"'] & \mathrm{Ind}\,\mathrm{D}_{\mathrm{cons}}((-)^{\mathrm{an}},\mathbb{Q})
\end{tikzcd}$$

where the pullback is taken in the ∞-category of functors $(\mathrm{Sch}^{\mathrm{ft}}_{\mathbb{C}})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{St}}$.

Because all categories have a perverse t-structure and that the functors are t-exact, the ∞-category $\mathrm{D}_{\mathrm{H}}(X,\mathbb{Z})$ affords a perverse t-structure, that restricts to compact objects, over any finite type $\mathbb{C}$-scheme X. Moreover, $\mathrm{D}_{\mathrm{H}}(-,\mathbb{Z})$ affords a 6-functor formalism.

Recall that in [Del71, Définition 2.3.1] Deligne defines mixed Hodge structures that have an integral lattice. In fact, we have a pullback for the abelian categories of polarisable mixed Hodge structures

$$\begin{tikzcd}
\mathrm{MHS}^P_{\mathbb{Z}}(\mathbb{C}) \arrow[r] \arrow[d, "\mathrm{int}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{MHS}^P_{\mathbb{Q}}(\mathbb{C}) \arrow[d] \\
\mathrm{Ab}^{\mathrm{ft}} \arrow[r] & \mathrm{Vect}^{\mathrm{fd}}_{\mathbb{Q}}
\end{tikzcd}$$

with $\mathrm{Vect}_{\mathbb{Q}}^{\mathrm{fd}}$ the abelian category of finite-dimensional $\mathbb{Q}$-vector spaces. (note that the weight filtration only exists with rational coefficients by definition) so that by the universal property of the fiber product, we obtain a functor

$$\mathrm{D}(\mathrm{Ind}\,\mathrm{MHS}_{\mathbb{Z}}^{P}(\mathbb{C})) \to \mathrm{D}_{\mathrm{H}}(\mathrm{Spec}(\mathbb{C})).$$

Proposition 6.1.2. *The above functor is an equivalence.*

Proof. This follows from the fact that the cohomological dimension of $\mathrm{MHS}_{\mathbb{Z}}^{P}$ is 1 by [Bei86, Corollary 1.10 and Remark 2.4]. Indeed in that case by [CD16, Lemma 1.1.7] any object of $\mathrm{D}^b(\mathrm{MHS}_{\mathbb{Z}}^{P})$ defined a compact object of $\mathrm{D}(\mathrm{Ind}\,\mathrm{MHS}_{\mathbb{Z}}^{P})$. ☐

There is a canonical realisation

$$\rho_{\mathrm{H}} \colon \mathrm{DM}^{\text{ét}}(-,\mathbb{Z}) \to \mathrm{D}_{\mathrm{H}}(-,\mathbb{Z})$$

that commutes with the 6 operations. Indeed, this functor is obtained from the universal property of the pullback, and as $\mathrm{D}_{\mathrm{H}}(-,\mathbb{Z})$ is compactly generated all operations commute with rationalisation so that the proof consist in proving that the exchange maps are equivalence when rationalised (this follows from [CD19, Theorem 4.2.29]) and when taken mod p for all primes p.

6.2 Mixed Hodge modules over $\mathbb{R}$-schemes.

In this section we extend the definition of mixed Hodge modules (with integral coefficients) to $\mathbb{R}$-schemes, using the same method as for Nori motives.

Let X be a finite type $\mathbb{R}$-scheme. Then $X_{\mathbb{C}} := X \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}(\mathbb{C})$ has a canonical $\mathbb{Z}/2\mathbb{Z}$ action given by the complex conjugation, and thus the ∞-category $\mathrm{D}_{\mathrm{H}}(X_{\mathbb{C}},\mathbb{Z})$ is endowed with a $\mathbb{Z}/2\mathbb{Z}$ action. In fact this construction is functorial and we obtain a functor

$$(\mathrm{Sch}_{\mathbb{R}}^{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{B}\mathbb{Z}/2\mathbb{Z}, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))$$

that sends a finite type $\mathbb{R}$-scheme X to $\mathrm{D}_{\mathrm{H}}(X_{\mathbb{C}},\mathbb{Z})$.

Now consider the functor

$$(-)^{\mathrm{h}\mathbb{Z}/2\mathbb{Z}} \colon \mathrm{Fun}(\mathrm{B}\mathbb{Z}/2\mathbb{Z}, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

that sends a presentably symmetric monoidal ∞-category $\mathcal{C}$ with a $\mathbb{Z}/2\mathbb{Z}$ action to the homotopy fixed points $\mathcal{C}^{\mathrm{h}\mathbb{Z}/2\mathbb{Z}}$. Note that this functor take a diagram $G \colon \mathrm{B}\mathbb{Z}/2\mathbb{Z} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ to its limit, thus has a left adjoint F that takes an object $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ to the constant diagram $\underline{\mathcal{C}}$.

Lemma 6.2.1. *If X is a finite type $\mathbb{C}$-scheme, then the natural functor*

$$p_1^* \colon \mathrm{D}_{\mathrm{H}}(X,\mathbb{Z}) \to \mathrm{D}_{\mathrm{H}}(X_{\mathbb{C}},\mathbb{Z})$$

induced by the first projection $p_1 \colon X \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \to X$ identifies naturally the left-hand side to the invariants of the right-hand side. More precisely there is a canonical isomorphism between the functor p_1^* and the counit

$$\eta \colon FG(\mathrm{D}_{\mathrm{H}}(X_{\mathbb{C}},\mathbb{Z})) \to \mathrm{D}_{\mathrm{H}}(X_{\mathbb{C}},\mathbb{Z}).$$

Proof. Indeed, one remarks that $X_{\mathbb{C}} \simeq X \times_{\mathbb{R}} X$, and that the projection map $X_{\mathbb{C}} \to X$ is an étale cover, so that the Bousfield-Kan formula for the limit of the $\mathbb{Z}/2\mathbb{Z}$-invariants is exactly the limit for étale descent. ☐

In particular the following definition does not introduce clash of notations:

Definition 6.2.2. The functoriality of the presentable ∞-category of mixed Hodge modules on finite type $\mathbb{R}$-schemes is the composition

$$\mathrm{D_H}(-,\mathbb{Z})\colon (\mathrm{Sch}_{\mathbb{R}}^{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{B}\mathbb{Z}/2\mathbb{Z}, \mathrm{CAlg}(\mathrm{Pr^L})) \xrightarrow{(-)^{\mathrm{h}\mathbb{Z}/2\mathbb{Z}}} \mathrm{CAlg}(\mathrm{Pr^L}).$$

By the previous lemma, the resriction of this functor to finite type $\mathbb{C}$-schemes corresponds to ind-mixed Hodge modules as constructed by M. Saito. We may denote by $\mathrm{D_H}(-,\mathbb{Q}) := \mathrm{D_H}(-,\mathbb{Z}) \otimes \mathrm{Mod}_{\mathbb{Q}}$.

Proposition 6.2.3. *Let X be a finite type $\mathbb{R}$-scheme. Then $\mathrm{D_H}(X,\mathbb{Q})$ is compactly generated and its full subcategory of compact objects is canonically isomorphic to $\mathrm{D}^b(\mathrm{MHM}(X,\mathbb{Q}))$ where $\mathrm{MHM}(X,\mathbb{Q})$ consists of the $\mathbb{Z}/2\mathbb{Z}$-invariants of $\mathrm{MHM}(X_{\mathbb{C}},\mathbb{Q})$.*

Proof. Because $\mathrm{D_H}(X_{\mathbb{C}},\mathbb{Q})$ has a t-structure that restricts to compact objects, it is true that $\mathrm{D_H}(X,\mathbb{Q})$ has a t-structure, that restricts to the invariants $\mathcal{D}(X)$ of the compact objects, seen as a full subcategory of $\mathrm{D_H}(X,\mathbb{Q})$. Now if D belongs to $\mathcal{D}(X)$, and $M = \mathrm{colim}_i\, M_i$ is a filtered colimit of objects of $\mathrm{D_H}(X,\mathbb{Q})$, we have that

$$\begin{aligned}
\mathrm{map}_{D_{\mathrm{H}}(X,\mathbb{Q})}(D, \mathrm{colim}\, M_i) &\simeq \mathrm{map}_{D_{\mathrm{H}}(X_{\mathbb{C}},\mathbb{Q})}(D_{|X_{\mathbb{C}}}, \mathrm{colim}_i (M_i)_{|X_{\mathbb{C}}})^{\mathrm{h}\mathbb{Z}/2\mathbb{Z}} \\
&\simeq \mathrm{R}\Gamma(\mathbb{R}_{\text{ét}}, \mathrm{colim}\, \mathrm{map}_{D_{\mathrm{H}}(X_{\mathbb{C}},\mathbb{Q})}(D_{|X_{\mathbb{C}}}, (M_i)_{|X_{\mathbb{C}}})) \\
&\overset{(*)}{\simeq} \mathrm{colim}_i\, \mathrm{R}\Gamma(\mathbb{R}_{\text{ét}}, \mathrm{map}_{D_{\mathrm{H}}(X_{\mathbb{C}},\mathbb{Q})}(D_{|X_{\mathbb{C}}}, (M_i)_{|X_{\mathbb{C}}})) \\
&\simeq \mathrm{colim}_i\, \mathrm{map}_{\mathrm{D_H}(X,\mathbb{Q})}(D, M_i)
\end{aligned}$$

where $(*)$ follows from [CD16, Lemma 1.1.10]. Thus any object of $\mathcal{D}(X)$ is compact. Moreover, because the functor

$$\mathrm{D_H}(X,\mathbb{Q}) \to \mathrm{D_H}(X_{\mathbb{C}},\mathbb{Q})$$

is conservative, the ∞-category $\mathcal{D}(X)$ generates $\mathrm{D_H}(X,\mathbb{Q})$ under colimits. Thus, the canonical functor $\mathrm{Ind}\,\mathcal{D}(X) \to \mathrm{D_H}(X,\mathbb{Z})$ is an equivalence.

Because p_1^* is perverse t-exact, the functor $\mathrm{MHM}(-,\mathbb{Q})$ has descent along p_1. Moreover, because $\mathrm{D}^b(\mathrm{MHM}(-,\mathbb{Q})) \to \mathcal{D}(-)$ is conservative and commutes with p_1^* as well as $(p_1)_*$ (which is also perverse t-exact), the same proof as [Tub23, Theorem 3.24] gives that $\mathrm{D}^b(\mathrm{MHM}(-,\mathbb{Q}))$ has descent for the morphism p_1, but this implies that the functor $\mathrm{D}^b(\mathrm{MHM}(X,\mathbb{Q})) \to \mathcal{D}(X)$ is an equivalence, finishing the proof. ☐

Corollary 6.2.4. *The functor $\mathrm{D_H}(-,\mathbb{Z})$ is an étale hypersheaf on finite type $\mathbb{R}$-schemes. For any finite type $\mathbb{R}$-scheme X, the ∞-category $\mathrm{D_H}(X,\mathbb{Z})$ affords a t-structure that restricts to the subcategory $\mathrm{D}^b_{\mathrm{H,c}}(X,\mathbb{Z})$of* constructible objects*, which consists of those objects whose underlying ind-constructible sheaf is constructible. We denote by $\mathrm{MHM}(X,\mathbb{Z})$ the heart of the latter ∞-category.*

Remark 6.2.5. Let X be a smooth $\mathbb{R}$-scheme. Let $M \in \mathrm{MHM}(X,\mathbb{Z})$, then $M_{|X_{\mathbb{C}}}$ admits an underlying $\mathcal{D}$-module on $X_{\mathbb{C}}$ with an action of $\mathbb{Z}/2\mathbb{Z}$, so that in fact we have a functor

$$\mathrm{MHM}(X,\mathbb{Z}) \to \mathcal{D}\text{-mod}(X)$$

by Galois descent of $\mathcal{D}$-modules.

Proposition 6.2.6. *The heart of* $\mathrm{D}^b_{\mathrm{H,c}}(\mathrm{Spec}(\mathbb{R}),\mathbb{Z})$ *is the category* $\mathrm{MHS}^p_{\mathbb{Z}}(\mathbb{R})$ *of polarisable mixed Hodge modules over* $\mathbb{R}$ *with integral coefficients. Moreover, the canonical functor*

$$\mathrm{D}^b(\mathrm{MHS}^p_{\mathbb{Z}}(\mathbb{R})) \to \mathrm{D}^b_{\mathrm{H,c}}(\mathrm{Spec}(\mathbb{R}),\mathbb{Z})$$

is an equivalence.

Proof. We can identify the heart with the abelian category of polarisable mixed Hodge structures over $\mathbb{C}$ with integral coefficients, together with an action of $\mathbb{Z}/2\mathbb{Z}$, and this is exactly $\mathrm{MHS}^p_{\mathbb{Z}}(\mathbb{R})$. Thus we have a functor

$$\mathrm{D}^b(\mathrm{MHS}^p_{\mathbb{Z}}(\mathbb{R})) \to \mathrm{D}^b_{\mathrm{H,c}}(\mathrm{Spec}(\mathbb{R}),\mathbb{Z}).$$

As in the proof of Proposition 6.2.3 the left-hand side has descent along the morphism $\mathbb{R} \to \mathbb{C}$, so that the result follows from Proposition 6.1.2. ☐

Corollary 6.2.7. *Let* $f\colon X \to \mathrm{Spec}\,\mathbb{R}$ *be a finite type map, then we recover the absolute Hodge cohomology groups of Beilinson ([Bei86]) as*

$$\mathrm{R}\Gamma_{\mathcal{H}}(X,\mathbb{Z}(p)) \simeq \mathrm{map}_{\mathrm{D_H}(\mathrm{Spec}(\mathbb{C}),\mathbb{Z})}(\mathbb{Z}, f_*\mathbb{Z}(p)) \simeq \mathrm{map}_{\mathrm{D_H}(X,\mathbb{Z})}(\mathbb{Z},\mathbb{Z}(p)).$$

Proof. Indeed Saito proved the rational result for $X_{\mathbb{C}}$ in [Sai90, Section 1.15], thus it suffices to check that Beilinson definition is indeed the $\mathbb{Z}/2\mathbb{Z}$-invariants of a pullback of the rational cohomology and the $\mathbb{Z}$-structure, but this is the case by [Bei86, Section 4.1, Theorem 3.4, Section 7]. ☐

We finish with a proposition computing the torsion of $\mathrm{D_H}(X,\mathbb{Z})$.

Proposition 6.2.8. *Let* X *be a finite type* $\mathbb{R}$*-scheme. Then the functor*

$$\mathrm{D}(X_{\text{ét}},\mathbb{Z}) \to \mathrm{D_H}(X,\mathbb{Z})$$

obtained by taking Galois invariant of the "Artin object functor"

$$\mathrm{D}((X_{\mathbb{C}})_{\text{ét}},\mathbb{Z}) \to \mathrm{DM}^{\text{ét}}(X_{\mathbb{C}},\mathbb{Z}) \to \mathrm{D_H}(X_{\mathbb{C}},\mathbb{Z})$$

induces an equivalence

$$\mathrm{D}(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z}) \simeq \mathrm{D_H}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}}.$$

Proof. Because everything is obtained by taking Galois invariants, it suffices to prove the claim when X is a finite type $\mathbb{C}$-scheme. In that case, the pullback definition provides us with a proof. ☐

6.3 Motivic Hodge modules

As for Nori motives, this realisation functor factors through modules over the algebra $\mathcal{H}$ representing Hodge cohomology. Using the same method as [Tub23, Section 4.3], we proved in [Tub24b, Section 2] that the canonical functor

$$\mathrm{Mod}_{\mathcal{H}\otimes_{\mathbb{Z}}\mathbb{Q}}(\mathrm{DM}^{\text{ét}}(-,\mathbb{Q})) \to \mathrm{D_H}(-)$$

is fully faithful over finite type $\mathbb{C}$-schemes, and that the t-structure on $\mathrm{D_H}(-)$ induces a t-structure on the image that we describe following Ayoub's ideas in [Ayo22, Theorem 1.98]: it is the localising subcategory generated by the $f_*\mathbb{Q}(n)$ with f a proper morphism and n an integer, so that we may call those *mixed Hodge modules of geometric origin* (this means that it is the smallest full subcategory stable under all colimits and finite limits that contains the $f_*\mathbb{Q}(n)$). This category of modules was first introduced by Drew in [Dre18].

There is also an enriched version of this result, where instead of obtaining the localising subcategory generated by the $f_*\mathbb{Q}(n)$, we obtain the localising subcategory generated by the $f_*H(n)$ for any mixed Hodge structure H (that we see as a constant mixed Hodge module), that we may call mixed Hodge modules of Hodge origin. Because this theory of enriched modules is more subtle than the geometric origin one, we will explain how to construct an integral version of the enriched category, and leave the case of geometric origin to the careful reader.

Construction 6.3.1. The functor $\mathrm{D_H}(-,\mathbb{Z})$ naturally takes values in $\mathrm{D}_H(\mathrm{Spec}(\mathbb{R}),\mathbb{Z})$-linear presentable ∞-categories. Thus, the functor ρ_{H} induces a $\mathrm{D}_H(\mathrm{Spec}(\mathbb{R}),\mathbb{Z})$-linear functor

$$\boldsymbol{\rho}_{\mathbf{H}}\colon \mathrm{DM}^{\text{ét}}(-,\mathbb{Z})\otimes_{\mathrm{Mod}_{\mathbb{Z}}}\mathrm{D}_H(\mathrm{Spec}(\mathbb{R}),\mathbb{Z})\to \mathrm{D_H}(-,\mathbb{Z}).$$

By the good properties of Lurie tensor product, it turns out that the left-hand side is

$$\mathbf{DM}^{\textbf{ét}}(-,\mathbb{Z}):=\mathrm{DM}^{\text{ét}}(-,\mathrm{D}_H(\mathrm{Spec}(\mathbb{R}),\mathbb{Z}))$$

the presentable ∞-category of étale motives with coefficients in mixed Hodge structures which has the same definition as étale motives with coefficient Λ, except that one replaces Mod_Λ by $\mathrm{D}_H(\mathrm{Spec}(\mathbb{R}),\mathbb{Z})$. The right adjoint $\boldsymbol{\rho}_*^{\mathbf{H}}$ of $\boldsymbol{\rho}_{\mathbf{H}}$ is lax monoidal thus we get an object $\boldsymbol{\mathcal{H}}_X=\boldsymbol{\rho}_*^{\mathbf{H}}(\mathbb{Z})$ of $\mathrm{CAlg}(\mathbf{DM}^{\textbf{ét}}(X,\mathbb{Z}))$ for all finite type $\mathbb{R}$-schemes X. There is a natural map

$$\pi_X^*\boldsymbol{\mathcal{H}}_{\mathrm{Spec}(\mathbb{C})}\to\boldsymbol{\mathcal{H}}_X$$

with $\pi_X\colon X\to\mathrm{Spec}(\mathbb{R})$ the structural morphism, it will be shown to be an isomorphism. The presentable ∞-categories of integral motivic Hodge modules is the category

$$\mathbf{DH}(X,\mathbb{Z}):=\mathrm{Mod}_{\pi_X^*\boldsymbol{\mathcal{H}}}(\mathbf{DM}^{\textbf{ét}}(X,\mathbb{Z})).$$

As for Nori motives, it is also the base change of $\mathbf{DM}^{\textbf{ét}}(-,\mathbb{Z})$ from $\mathbf{DM}^{\textbf{ét}}(\mathrm{Spec}(\mathbb{R}),\mathbb{Z})$-linear categories to $\mathbf{DH}(\mathrm{Spec}(\mathbb{R}),\mathbb{Z})$-linear categories:

$$\mathbf{DH}(-,\mathbb{Z})\simeq\mathbf{DM}^{\textbf{ét}}(-,\mathbb{Z})\otimes_{\mathbf{DM}^{\textbf{ét}}(\mathrm{Spec}(\mathbb{R}),\mathbb{Z})}\mathbf{DH}(\mathrm{Spec}(\mathbb{R}),\mathbb{Z}).$$

Thus as in Theorem 3.3.1, the functor $\mathbf{DH}(-,\mathbb{Z})$ affords all the six operations in such a way that the functor from $\mathbf{DM}^{\textbf{ét}}(-,\mathbb{Z})$ and the canonical functor to $\mathrm{D_H}(-,\mathbb{Z})$ commute with them, the composition of the two being $\boldsymbol{\rho}_{\mathbf{H}}$.

Lemma 6.3.2. *Let* $\iota\colon\mathrm{Spec}(\mathbb{C})\to\mathrm{Spec}(\mathbb{R})$ *be the spectrum of the inclusion of* $\mathbb{R}$ *into* $\mathbb{C}$*. Then the canonical map* $\iota^*\boldsymbol{\mathcal{H}}_{\mathrm{Spec}(\mathbb{R})}\to\boldsymbol{\mathcal{H}}_{\mathrm{Spec}(\mathbb{C})}$ *is an equivalence.*

Proof. Becuase ι is finite étale, the functor $\iota^*=i^!$, being a right adjoint, commutes with $\boldsymbol{\rho}_{\mathbf{H}}$. Thus $\iota^*\boldsymbol{\mathcal{H}}_{\mathrm{Spec}(\mathbb{R})}\simeq\boldsymbol{\rho}_{\mathbf{H}}(\iota^*\mathbb{1})\simeq\boldsymbol{\mathcal{H}}_{\mathrm{Spec}(\mathbb{C})}$. □

Theorem 6.3.3. *The canonical functor*

$$\underline{\boldsymbol{\rho}_{\mathbf{H}}}\colon\mathbf{DH}(-,\mathbb{Z})\to\mathrm{D_H}(-,\mathbb{Z})$$

factoring $\boldsymbol{\rho}_{\mathbf{H}}$ *is fully faithful and its image is stable under truncations.*

Proof. By étale descent and Lemma 6.3.2, it suffices to prove the claim over finite type $\mathbb{C}$-schemes. Because $\mathrm{D}_{\mathrm{H}}(-,\mathbb{Z})$ is compactly generated, the right adjoint $\boldsymbol{\rho}_*^{\mathbf{H}}$ commutes with rationalisation, so that the algebra $\pi_X^*\boldsymbol{\mathcal{H}}_{\mathrm{Spec}(\mathbb{C})}$, tensored with $\mathbb{Q}$ is the algebra considered by Drew in [Dre18] and the second author in [Tub24b]. This implies that $\underline{\boldsymbol{\rho}_{\mathbf{H}}} \otimes \mathrm{Mod}_{\mathbb{Q}}$ is fully faithful with image stable under truncations by the main theorem of [Tub24b, Section 2] (see also[Ayo22, Theorem 1.98]). When tensoring the square of Definition 6.1.1 by $\mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}}$, we obtain an equivalence

$$\mathrm{D}_{\mathrm{H}}(-,\mathbb{Z}) \otimes_{\mathrm{Mod}_{\mathbb{Z}}} \mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}} \xrightarrow{\sim} \mathrm{Ind}(\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{an}},\mathbb{Z}) \otimes_{\mathrm{Perf}_{\mathbb{Z}}} \mathrm{Perf}_{\mathbb{Z}/n\mathbb{Z}}),$$

but the right-hand side coincides with the derived category of étale sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on X, thanks to Artin comparison theorem. By rigidity, this implies that the natural map

$$\mathbb{Z}/n\mathbb{Z} \to \pi_X^*\boldsymbol{\mathcal{H}}_{\mathrm{Spec}(\mathbb{C})}/n$$

is an equivalence, and then that $\underline{\boldsymbol{\rho}_{\mathbf{H}}} \otimes_{\mathrm{Mod}_{\mathbb{Z}}} \mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}}$ is an equivalence. This finishes the proof of the full faithfulness.

Now let M be in the image of $\underline{\boldsymbol{\rho}_{\mathbf{H}}}$. Because the t-structure is compatible with filtered colimits (this is the case for the t-structure of all categories in the pullback defining D_{H}), we have an exact triangle

$$\tau^{\leqslant 0}M \to \tau^{\leqslant 0}(M \otimes_{\mathbb{Z}} \mathbb{Q}) \to \mathrm{colim}_n\, \tau^{\leqslant 0}(M) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}.$$

The middle term is in the image because we proved the theorem with rational coefficients in [Tub24b]. Thus it suffices to show that for all $n \in \mathbb{N}^*$, the object $\tau^{\leqslant 0}(M) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is in the image. This is the case because $\underline{\boldsymbol{\rho}_{\mathbf{H}}} \otimes_{\mathrm{Mod}_{\mathbb{Z}}} \mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}}$ is an equivalence. This proves that the image is stable under truncations, finishing the proof. □

Remark 6.3.4. The fact that $\underline{\boldsymbol{\rho}_{\mathbf{H}}}$ is fully faithful implies readily that the map $\pi_X^*\boldsymbol{\mathcal{H}}_{\mathrm{Spec}(\mathbb{R})} \to \boldsymbol{\mathcal{H}}_X$ is an equivalence for all X.

Remark 6.3.5. As explained at the beginning of this subsection, the above theorem also works for the non enriched version of the Hodge realisation of étale motives. We will call those Hodge modules of geometric origin, and denote them by $\mathrm{DH}(-,\mathbb{Z})$.

Remark 6.3.6. For X a finite type $\mathbb{R}$-scheme, the image of $\boldsymbol{\rho}_{\mathbf{H}}$ can be described as the localising subcategory spanned by objects of the form $p_*H(n)$ for $p\colon Y \to X$ proper, and H a polarisable integral mixed Hodge structure over $\mathbb{R}$ seen as a constant integral mixed Hodge module over Y.

6.4 Abelian categories of integral Hodge modules

By definition and by Theorem 6.3.3 for each finite type $\mathbb{R}$-scheme X the three categories $\mathrm{D}_{\mathrm{H}}(X,\mathbb{Z})$, $\mathrm{DH}(X,\mathbb{Z})$ and $\mathbf{DH}(X,\mathbb{Z})$ admit a perverse t-structure, that restrict to constructible object $\mathrm{D}^b_{\mathrm{H,c}}(X,\mathbb{Z})$, $\mathrm{DH}_c(X,\mathbb{Z})$ and $\mathbf{DH}_c(X,\mathbb{Z})$. As above, the two categories $\mathrm{DH}_c(X,\mathbb{Z})$ and $\mathbf{DH}_c(X,\mathbb{Z})$ are the thick categories spanned by the generators of Remark 6.3.6.

We will denote by $\mathrm{MHM}(X,\mathbb{Z})$, $\mathrm{MHM}_{\mathrm{geo}}(X,\mathbb{Z})$ and $\mathrm{MHM}_{\mathrm{Hdg}}(X,\mathbb{Z})$ the three hearts of the constructible objects.

In more concrete terms, a mixed Hodge module with integral coefficients is the data of three objects $(M,\mathcal{F},S,\varphi)$ where $M \in \mathrm{MHM}(X_{\mathbb{C}},\mathbb{Q})$ is a mixed Hodge module as constructed by M. Saito, $\mathcal{F} \in \mathrm{Perv}(X_{\mathbb{C}}^{\mathrm{an}},\mathbb{Z})$ is a perverse sheaf, $S \in \mathrm{End}(M,\mathcal{F})$ is an involution and $\varphi\colon \mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{rat}(M)$ is an isomorphism of perverse sheaves that commutes with S.

Because we have the 6 operations, we can also construct ordinary t-structures on the constructible objects by gluing along stratifications, doing the same construction as in the proof of Proposition 5.1.1 in reverse. We will denote their hearts by $\mathrm{MHM}^{\mathrm{ord}}(X,\mathbb{Z})$, $\mathrm{MHM}^{\mathrm{ord}}_{\mathrm{geo}}(X,\mathbb{Z})$ and $\mathrm{MHM}^{\mathrm{ord}}_{\mathrm{Hdg}}(X,\mathbb{Z})$ respectively. Using the same proof as Theorem 4.3.2, we obtain:

Proposition 6.4.1. *The natural functors*

$$\mathrm{D}^b(\mathrm{MHM}^{\mathrm{ord}}(X,\mathbb{Z})) \to \mathrm{D}^b_{\mathrm{H,c}}(X,\mathbb{Z}),$$

$$\mathrm{D}^b(\mathrm{MHM}^{\mathrm{ord}}_{\mathrm{geo}}(X,\mathbb{Z})) \to \mathrm{DH}_c(X,\mathbb{Z})$$

and

$$\mathrm{D}^b(\mathrm{MHM}^{\mathrm{ord}}_{\mathrm{Hdg}}(X,\mathbb{Z})) \to \mathbf{DH}_c(X,\mathbb{Z})$$

are equivalences.

Corollary 6.4.2. *Over finite type $\mathbb{C}$-schemes, the natural functors*

$$\mathrm{D}(\operatorname{Ind}\mathrm{MHM}^{\mathrm{ord}}(X,\mathbb{Z})) \to \mathrm{D}_{\mathrm{H}}(X,\mathbb{Z}),$$

$$\mathrm{D}(\operatorname{Ind}\mathrm{MHM}^{\mathrm{ord}}_{\mathrm{geo}}(X,\mathbb{Z})) \to \mathrm{DH}(X,\mathbb{Z})$$

and

$$\mathrm{D}(\operatorname{Ind}\mathrm{MHM}^{\mathrm{ord}}_{\mathrm{Hdg}}(X,\mathbb{Z})) \to \mathbf{DH}(X,\mathbb{Z})$$

are equivalences.

Proof. Indeed the hearts of the compact objects have finite cohomological dimension (by the same argument as for the perverse hearts), so that the corollary follows by taking indization of the theorem. ☐

Remark 6.4.3. The above result is false for ordinary motives over $\mathbb{C}$-schemes. Indeed, assuming that the functor $\mathrm{DM}^{\text{ét}}(X,\Lambda) \to \mathrm{DN}(X,\Lambda)$ is an equivalence, Ayoub's counterexample [Ayo17, Lemma 2.4] would imply that the functor $\mathrm{DN}(X,\Lambda) \to \mathrm{D}(X^{\mathrm{an}},\Lambda)$ is not conservative, but the functor $\mathrm{D}(\operatorname{Ind}\mathcal{M}_{\mathrm{ord}}(X,\Lambda)) \to \mathrm{D}(X^{\mathrm{an}},\Lambda)$ *is* conservative. This is the only obstruction, see Proposition 2.2.3.

Remark 6.4.4. As for Nori motives, we suspect that the functors

$$\mathrm{D}^b(\mathrm{MHM}_{\mathrm{geo}}(X,\mathbb{Z})) \to \mathrm{DH}_c(X,\mathbb{Z})$$

and

$$\mathrm{D}^b(\mathrm{MHM}_{\mathrm{Hdg}}(X,\mathbb{Z})) \to \mathbf{DH}_c(X,\mathbb{Z})$$

are equivalences, because in some sense, objects of geometric origin have a tendency to be flat, or at least, resolved by flat objects.

We finish the paper by linking our notion of mixed Hodge modules to the classical notion of *variation of mixed Hodge structure*. We say that an integral mixed Hodge module $M \in \mathrm{MHM}(X,\mathbb{Z})$ is lisse if its underlying perverse sheaf $\mathrm{int}(M)$ is locally constant. We denote by $\mathrm{MHM}_{\mathrm{lis}}(X,\mathbb{Z})$ the category of lisse Mixed Hodge modules on a finite-type $\mathbb{C}$-scheme. Note that the pullback defining $\mathrm{D}_{\mathrm{H}}(X,\mathbb{Z})$ implies that the perverse t-structure induces a t-structure on the subcategory of dualisable objects of $\mathrm{D}_{\mathrm{H}}(X,\mathbb{Z})$; the heart of this t-structure is precisely $\mathrm{MHM}_{\mathrm{lis}}(X,\mathbb{Z})$.

Proposition 6.4.5. *Let X be a finite-type $\mathbb{C}$-scheme and let $\mathrm{VMHS}_{\mathrm{ad}}(X,\mathbb{Z})$ be the category of admissible variations of mixed Hodge structures, where a variation of mixed Hodge structure in the sense of [Del80, 1.8.14] is admissible if its rationalisation is admissible in the sense of [Sai89, 2.1]. Then, we have an equivalence of categories*

$$\mathrm{MHM}_{\mathrm{lis}}(X,\mathbb{Z}) \to \mathrm{VMHS}_{\mathrm{ad}}(X,\mathbb{Z}).$$

Proof. We have a pullback square:

$$\begin{tikzcd}
\mathrm{MHM}_{\mathrm{lis}}(X, \mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{Loc}(X^{\mathrm{an}}, \mathbb{Z}) \arrow[d] \\
\mathrm{MHM}_{\mathrm{lis}}(X, \mathbb{Q}) \arrow[r] & \mathrm{Loc}(X^{\mathrm{an}}, \mathbb{Q})
\end{tikzcd}$$

where $\mathrm{Loc}(X, \Lambda)$ is the category of locally constant sheaves of Λ-modules with values a in finitely generated Λ-modules.

By [Sai89, Theorem 2.2], we have an equivalence of categories

$$\mathrm{MHM}_{\mathrm{lis}}(X, \mathbb{Q}) \to \mathrm{VMHS}_{\mathrm{ad}}(X, \mathbb{Q}).$$

But by definition, a VMHS with integral coefficients is exactly a VMHS with rational coefficients admitting an integral lattice. Hence, we also have a pullback diagram:

$$\begin{tikzcd}
\mathrm{VMHS}_{\mathrm{ad}}(X, \mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{Loc}(X^{\mathrm{an}}, \mathbb{Z}) \arrow[d] \\
\mathrm{VMHS}_{\mathrm{ad}}(X, \mathbb{Q}) \arrow[r] & \mathrm{Loc}(X^{\mathrm{an}}, \mathbb{Q}),
\end{tikzcd}$$

and the result follows. ☐

Remark 6.4.6. One could go further and define the usual operations of mixed Hodge modules, but with integral coefficients. For example if $f\colon X \to \mathbb{A}^1_{\mathbb{C}}$ is a map of finite type $\mathbb{C}$-schemes, one can define nearby cycles Ψ_f associated to f as the functor

$$\Psi_f\colon \mathrm{D}^b_{\mathrm{H,c}}(X_\eta, \mathbb{Z}) \to \mathrm{D}^b_{\mathrm{H,c}}(X_s, \mathbb{Z}),$$

with $X_\eta = f^{-1}(\mathbb{G}_m)$ and $X_s = f^{-1}(\{0\})$, defined by the pullback property:

$$\begin{tikzcd}
\mathrm{D}_{\mathrm{H,c}}(X_\eta, \mathbb{Z}) \arrow[dr, dashed, "\Psi_f"] \arrow[drr, bend left, "\Psi_f(\mathrm{int}(-))"] \arrow[ddr, bend right, "\Psi_f(-\otimes_{\mathbb{Z}}\mathbb{Q})"'] & & \\
& \mathrm{D}_{\mathrm{H,c}}(X_s, \mathbb{Z}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{D}^b_c(X_s, \mathbb{Z}) \arrow[d] \ , \\
& \mathrm{D}_{\mathrm{H,c}}(X_s, \mathbb{Q}) \arrow[r] & \mathrm{D}^b_c(X_s, \mathbb{Z})
\end{tikzcd}$$

so that all known properties about Ψ_f on $\mathrm{D}^b_c(-, \mathbb{Z})$ remain true for the one for $\mathrm{D}^b_{\mathrm{H,c}}(-, \mathbb{Z})$, for example t-exactness or compatibility with duality. Similarly, one can define vanishing cycles, monodromic mixed Hodge modules and have a Thom-Sebastiani formula by pullback. Taking $\mathbb{Z}/2\mathbb{Z}$-invariants, we also have a version over finite type $\mathbb{R}$-schemes.

References

[AC23] Dhyan Aranha and Chirantan Chowdhury. The chow weight structure for geometric motives of quotient stacks, 2023. Preprint, available at http://arxiv.org/abs/2306.10557.

[AGV22] Joseph Ayoub, Martin Gallauer, and Alberto Vezzani. The six-functor formalism for rigid analytic motives. *Forum Math. Sigma*, 10:Paper No. e61, 182, 2022.

[Ara05] Donu Arapura. The Leray spectral sequence is motivic. *Invent. Math.*, 160(3):567–589, 2005.

[Ara13] Donu Arapura. An abelian category of motivic sheaves. *Adv. Math.*, 233:135–195, 2013.

[Ayo07] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II. *Astérisque*, (315):x+370, 2007.

[Ayo10] Joseph Ayoub. Note sur les opérations de Grothendieck et la réalisation de Betti. *J. Inst. Math. Jussieu*, 9(2):225–263, 2010.

[Ayo14] Joseph Ayoub. La réalisation étale et les opérations de Grothendieck. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(1):1–145, 2014.

[Ayo17] Joseph Ayoub. Motives and algebraic cycles: a selection of conjectures and open questions. In *Hodge theory and L^2-analysis*, volume 39 of *Adv. Lect. Math. (ALM)*, pages 87–125. Int. Press, Somerville, MA, 2017.

[Ayo22] Joseph Ayoub. Anabelian presentation of the motivic Galois group in characteristic zero. Preprint, available at `https://user.math.uzh.ch/ayoub/PDF-Files/Anabel.pdf`, 2022.

[Bac21] Tom Bachmann. Rigidity in étale motivic stable homotopy theory. *Algebr. Geom. Topol.*, 21(1):173–209, 2021.

[BBD82] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[BCKW19] Ulrich Bunke, Denis-Charles Cisinski, Daniel Kasprowski, and Christoph Winges. Controlled objects in left-exact ∞-categories and the Novikov conjecture. Preprint, available at `https://cisinski.app.uni-regensburg.de/unik.pdf`, 2019.

[Bei86] Alexander A. Beilinson. Notes on absolute Hodge cohomology. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 35–68. Amer. Math. Soc., Providence, RI, 1986.

[Bei87] Alexander A. Beilinson. On the derived category of perverse sheaves. In *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, volume 1289 of *Lecture Notes in Math.*, pages 27–41. Springer, Berlin, 1987.

[BGH20] Clark Barwick, Saul Glasman, and Peter Haine. Exodromy, 2020.

[BGT14] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. Uniqueness of the multiplicative cyclotomic trace. *Adv. Math.*, 260:191–232, 2014.

[BH21] Tom Bachmann and Marc Hoyois. Remarks on étale motivic stable homotopy theory. Preprint, available at `https://arxiv.org/abs/2104.06002`, 2021.

[BM21] Bhargav Bhatt and Akhil Mathew. The arc-topology. *Duke Math. J.*, 170(9):1899–1988, 2021.

[BM24] Shay Ben-Moshe. Naturality of the ∞-categorical enriched Yoneda embedding. *J. Pure Appl. Algebra*, 228(6):Paper No. 107625, 25, 2024.

[BS15] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. In *De la géométrie algébrique aux formes automorphes (I). Une collection d'articles en l'honneur du soixantième anniversaire de Gérard Laumon*, pages 99–201. Paris: Société Mathématique de France (SMF), 2015.

[BVK16] Luca Barbieri-Viale and Bruno Kahn. On the derived category of 1-motives. *Astérisque*, (381):xi+254, 2016.

[CD12] Denis-Charles Cisinski and Frédéric Déglise. Mixed Weil cohomologies. *Adv. Math.*, 230(1):55–130, 2012.

[CD16] Denis-Charles Cisinski and Frédéric Déglise. Étale motives. *Compos. Math.*, 152(3):556–666, 2016.

[CD19] Denis-Charles Cisinski and Frédéric Déglise. *Triangulated categories of mixed motives.* Springer Monographs in Mathematics. Springer, Cham, 2019.

[CG17] Utsav Choudhury and Martin Gallauer. An isomorphism of motivic Galois groups. *Adv. Math.*, 313:470–536, 2017.

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra*, volume 180 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2019.

[Del68] Pierre Deligne. Théorème de Lefschetz et critères de dégénérescence de suites spectrales. *Publ. Math., Inst. Hautes Étud. Sci.*, 35:107–126, 1968.

[Del71] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[Del80] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[Del90] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 111–195. Birkhäuser Boston, Boston, MA, 1990.

[DG22] Brad Drew and Martin Gallauer. The universal six-functor formalism. *Ann. K-Theory*, 7(4):599–649, 2022.

[Dre18] Brad Drew. Motivic Hodge modules, 2018. Preprint, available at https://bdrew.gitlab.io/pdf/DH.pdf.

[EHIK21a] Elden Elmanto, Marc Hoyois, Ryomei Iwasa, and Shane Kelly. Cdh descent, cdarc descent, and Milnor excision. *Math. Ann.*, 379(3-4):1011–1045, 2021.

[EHIK21b] Elden Elmanto, Marc Hoyois, Ryomei Iwasa, and Shane Kelly. Milnor excision for motivic spectra. *J. Reine Angew. Math.*, 779:223–235, 2021.

[Fak00] Najmuddin Fakhruddin. Notes of Nori's lectures on Mixed Motives. TIFR, Mumbai,Preprint, 2000.

[GM83] Mark Goresky and Robert MacPherson. Intersection homology. II. *Invent. Math.*, 72(1):77–129, 1983.

[Gro66] Alexander Grothendieck. Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas. (Troisième partie). Rédigé avec la colloboration de J. Dieudonné. *Publ. Math., Inst. Hautes Étud. Sci.*, 28:1–255, 1966.

[GS83] Mark Goresky and Paul Siegel. Linking pairings on singular spaces. *Comment. Math. Helv.*, 58(1):96–110, 1983.

[Hai22] Peter J. Haine. From nonabelian basechange to basechange with coefficients. Preprint, available at `http://arxiv.org/abs/2108.03545`, 2022.

[Har16] Daniel Harrer. Comparison of the Categories of Motives defined by Voevodsky and Nori, 2016. Phd thesis, available at `http://arxiv.org/abs/1609.05516`.

[Hau21] Rune Haugseng. On lax transformations, adjunctions, and monads in $(\infty, 2)$-categories. *High. Struct.*, 5(1):244–281, 2021.

[HL23] Quoc P. Ho and Penghui Li. Revisiting mixed geometry. Preprint, available at `http://arxiv.org/abs/2202.04833`, 2023.

[HMS17] Annette Huber and Stefan Müller-Stach. *Periods and Nori motives*, volume 65 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2017. With contributions by Benjamin Friedrich and Jonas von Wangenheim.

[HRS23] Tamir Hemo, Timo Richarz, and Jakob Scholbach. Constructible sheaves on schemes. *Adv. Math.*, 429:Paper No. 109179, 46, 2023.

[HS23] David Hansen and Peter Scholze. Relative perversity. *Comm. Amer. Math. Soc.*, 3:631–668, 2023.

[HSS17] Marc Hoyois, Sarah Scherotzke, and Nicolò Sibilla. Higher traces, noncommutative motives, and the categorified Chern character. *Adv. Math.*, 309:97–154, 2017.

[Hub93] Annette Huber. Calculation of derived functors via Ind-categories. *J. Pure Appl. Algebra*, 90(1):39–48, 1993.

[ILO14] Luc Illusie, Yves Laszlo, and Fabrice Orgogozo, editors. *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents. Séminaire à l'École polytechnique 2006-2008.* Number 363-364 in Astérisque. Société mathématique de France, 2014.

[IM24] Florian Ivorra and Sophie Morel. The four operations on perverse motives. *J. Eur. Math. Soc.*, 2024. published online first.

[Ivo16] Florian Ivorra. Perverse, Hodge and motivic realizations of étale motives. *Compos. Math.*, 152(6):1237–1285, 2016.

[Ivo17] Florian Ivorra. Perverse Nori motives. *Math. Res. Lett.*, 24(4):1097–1131, 2017.

[Iwa18] Isamu Iwanari. Tannaka duality and stable infinity-categories. *J. Topol.*, 11(2):469–526, 2018.

[Kra05] Henning Krause. The stable derived category of a noetherian scheme. *Compos. Math.*, 141(5):1128–1162, 2005.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[HA] Jacob Lurie. Higher Algebra, 2022. Available at `https://people.math.harvard.edu/~lurie/papers/HA.pdf`.

[SAG] Jacob Lurie. Spectral Algebraic Geometry. Preprint, available at `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`.

[Nor02] Madhav V. Nori. Constructible sheaves. In *Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000)*, volume 16 of *Tata Inst. Fund. Res. Stud. Math.*, pages 471–491. Tata Inst. Fund. Res., Bombay, 2002.

[NV15] Arvind N. Nair and Vaibhav Vaish. Weightless cohomology of algebraic varieties. *J. Algebra*, 424:147–189, 2015.

[Pri20] Jonathan P. Pridham. Tannaka duality for enhanced triangulated categories I: reconstruction. *J. Noncommut. Geom.*, 14(2):591–637, 2020.

[Rob14] Marco Robalo. *Théorie homotopique motivique des espaces non commutatifs*. PhD thesis, Université de Montpellier 2, 2014.

[RS20] Timo Richarz and Jakob Scholbach. The intersection motive of the moduli stack of shtukas. *Forum Math. Sigma*, 8:Paper No. e8, 99, 2020.

[RT24] Raphaël Ruimy and Swann Tubach. Étale motives of geometric origin. Preprint, available at `https://arxiv.org/abs/2405.07095`, 2024.

[Rui22] Raphaël Ruimy. Integral Artin motives: t-structures and Artin Vanishing Theorem, 2022. Preprint, available at `https://arxiv.org/abs/2211.02505`.

[Rui24] Raphaël Ruimy. Artin perverse sheaves. *J. Algebra*, 639:596–677, 2024.

[Sai89] Morihiko Saito. Introduction to mixed Hodge modules. In *Actes du colloque de théorie de Hodge. Luminy, France, 1-6 Juin 1987*, pages 145–162. Paris: Société Mathématique de France, 1989.

[Sai90] Morihiko Saito. Mixed Hodge modules. *Publ. Res. Inst. Math. Sci.*, 26(2):221–333, 1990.

[SGA4] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Dirigé par Michael Artin, Alexander Grothendieck, et Jean-Louis. Verdier. Avec la collaboration de Nicolas Bourbaki, Pierre Deligne et Bernard Saint-Donat.

[Ter24] Luca Terenzi. Tensor structure on perverse Nori motives, 2024. Preprint, available at http://arxiv.org/abs/2401.13547.

[Tub23] Swann Tubach. On the Nori and Hodge realisations of Voevodsky étale motives. Preprint, available at https://swann.tubach.fr/en/research/, 2023.

[Tub24a] Swann Tubach. Artin motives in relative Nori and Voevodsky motives. Preprint, available at http://swann.tubach.fr, 2024.

[Tub24b] Swann Tubach. Mixed Hodge modules on stacks. Preprint, available at https://swann.tubach.fr/en/research/, 2024.

[Vai20] Vaibhav Vaish. Punctual gluing of t-structures and weight structures. *Manuscripta Math.*, 162(3-4):341–366, 2020.

[Wil12] Jörg Wildeshaus. Motivic intersection complex. In *Regulators*, volume 571 of *Contemp. Math.*, pages 255–276. Amer. Math. Soc., Providence, RI, 2012.

[Wil19] Jörg Wildeshaus. On the intersection motive of certain Shimura varieties: the case of Siegel threefolds. *Ann. K-Theory*, 4(4):525–559, 2019.

Raphaël Ruimy raphael.ruimy@unimi.it
Dipartimento di Matematica "F. Enriques" - Universita degli Studi di Milano,
via Cesare Saldini 50, I-20133 Milano, Italy

Swann Tubach swann.tubach@ens-lyon.fr
E.N.S Lyon, UMPA,
46 Allée d'Italie, 69364 Lyon Cedex 07, France